

# Three Kantian Strands in Frege's View of Arithmetic

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**Abstract** - On the background of explaining their different notions of analyticity (displayed, e.g. by the above), their different views on definitions, and some aspects of Frege's notion of sense, three important Kantian strands that interweave into Frege's view are exposed. First, Frege's remarkable view that arithmetic, though analytic, contains truths that "extend our knowledge", and by Kant's use of the term, should be regarded synthetic. Secondly, that our arithmetical (and logical) knowledge depends on a sort of *Anschauung* – a capacity to recognize and identify objects, which are given us in particular ways, constituting their senses (*Sinne*). Third, that there is a sense in which Frege's view of definitions and explications gives new substance to Kant's leading idea of analyticity, namely, the containment of a truth or a concept in another. In all these, Frege's view does not endorse the Kantian strands as they are, but gives them special and sometimes quite sophisticated twists.

One problem confronting the notion that arithmetic contains genuine knowledge is its analyticity: if arithmetic is, as Frege famously proclaimed, analytic,<sup>1</sup> doesn't this mean – as Kant and many of the positivists thought – that it does not express genuine objective knowledge?<sup>2</sup> Frege's answer, as we shall see, is No. His position here is quite remarkable in holding both that arithmetic is analytic and that it contains genuine objective knowledge and "extends our knowledge". In the course of elaborating this I shall indicate the difference between Kant's and Frege's notions of analyticity, yet try to elucidate three Kantian strands<sup>3</sup> in Frege's view:

1. Arithmetic, though analytic, extends our knowledge, and in Kant's sense of the term is synthetic.
2. Arithmetical (and logical) knowledge depends on our capacity to recognize objects.
3. The reduction of arithmetic to logic (via appropriate definitions) displays a version of Kant's leading idea of analyticity – the idea of conceptual containment.

On the relationships between Kant's and Frege's views on mathematics, there is a standard story going somewhat like this: Frege was a Kantian on Geometry, accepting

Kant's view that geometry is synthetic a priori, but rejected Kant's view that arithmetic is synthetic, and argued, or even proved, that arithmetic is analytic.<sup>4</sup>

There are some misleading points in this formulation. Frege's view of arithmetic would amount to rejecting Kant's if they were using "analytic" and "synthetic" in the same way. But they were *not*. First, though the point has been debated in Kant's scholarship, most commentators concede that for Kant, analyticity is a property of the *content* of a proposition (or a judgment), namely, that its subject (-concept) contains its predicate (-concept)<sup>5</sup>; for Frege, it concerns its *justifiability*, namely, whether it can be justified on the basis of logic and definitions alone (more on this in the sequel). Though there is obviously a connection between the two, there are also important differences. And secondly, Kant's notion, besides being too restrictive in applying only to subject-predicate propositions, is notoriously hazy and unclear, as the relevant notions of subject, predicate and containment are, in contrast to Frege's wider and much more precise one. In fact, at least in Frege's eyes, the differences are so significant that, as we shall see in the sequel, Frege explicitly says in FA that by Kant's notions of analytic and synthetic, arithmetic should be deemed synthetic!

Likewise, whether or not arithmetic is justifiable on the basis of logic and definitions depends on the nature and scope of logic and on those of the acceptable definitions. Evidently, Frege's and Kant's ideas on these differ substantially. Many scholars argued that the differences in their views of logic are so vast and basic that a serious debate on the nature of arithmetic and its reducibility to logic is unintelligible. Others contested this view and argued that the great differences in their views on logic notwithstanding, there still is a shared basic core of their conceptions of the essence of logic, which makes a serious debate here possible – both on the scope of logic and on the nature of arithmetic and the logicistic thesis<sup>6</sup>.

I shall leave aside here the nature of logic and the important differences in Kant's and Frege's views of logic and its scope, and claim that even assuming that Kant would, or should have accepted Frege's logic, there is much amiss in the standard story, and the above difference in their notions of analyticity (apriority etc.) is decisive. But yet, looked more deeply, **Frege's conception of arithmetic has important Kantian strands**,<sup>7</sup> where a key to understanding this is to note some aspects of his notion of sense, which have been rather ignored or played down in the literature. I should emphasize that my main concern in this paper is with Frege's view, not Kant's. I shall therefore allow myself to

remain quite general and imprecise about Kant. This, I hope, should not be too detrimental to my detecting general Kantian strands in Frege's view. On this background it may be interesting to examine various interpretations of these strands in Kant and their effect on the theses proposed here, but this would go beyond the scope of this paper.

### **SENSE AND THE JUSTIFICATION OF AXIOMS**

The notion of sense is central in Frege's philosophy, even in his writings before he introduced his systematic terminological distinction between sense (*Sinn*) and reference (*Bedeutung*) in the early nineties. Sense is primarily "mode of being given" or "mode of presentation" - *Art des Gegebenseins* – in which something (an object, or function, including concepts) is given to us as the reference of a term.<sup>8</sup> The phrase is, of course, Kantian. But the differences between them notwithstanding, it is not only the term but the general conception that things in the world are always given to us in particular ways (waving for the moment the nature of these ways), which Frege inherited from Kant. Yet, he gave it a sort of a linguistic turn: A sense, for Frege, is a mode in which something (an object or a function) is being given to us, as this is expressed in the meaning of a linguistic expression referring to it. Hence, his conception of sense is heavily constrained by his elaborate theory of reference, on which it supervenes. Thus conceived, sense is yet an epistemic notion, where the cognitive value of statements lies.

All this has been much discussed.<sup>9</sup> What has been less discussed is a certain aspect of the notion of sense - its being a *justificatory* one: it lies, as I argue, at the basis of the justificatory enterprise in *justifying the axioms*, or the basic truths of a domain. And this is vital for Frege's notion of analyticity and for appreciating his view that arithmetic, though extending our knowledge, is still analytic. The issues concerned revolve around three main claims: 1) The notions of analytic and a priori apply only to justifications of propositions. 2) Axioms (basic laws) of logic are analytic, and those of geometry – a priori. 3) Deductive derivation (from truths) is a basic and paradigmatic form of justification, but when it comes to the axioms, which are not derivable from other truths, justification can take other forms. All three raise serious problems and have been challenged. Though I cannot discuss them in detail here, let me expand on them a bit.

Though the point has been quite surprisingly ignored or played down in much of the secondary literature, Frege is clear, in introducing the notion of analyticity in §3 of *Foundations of Arithmetic* (FA), about its epistemic and justificatory nature.<sup>10</sup> A proposition is analytic, according to Frege, "if in carrying out this process [of finding a

proof of it and following it up back to the primitive truths] we come only on general logical laws and on definitions" (FA, §3, p. 4). But this is not only implied by his "definition" of analyticity, but stated clearly as a governing principle:

**"When a proposition is called a posteriori or analytic in my sense [...] it is a judgment about the ultimate ground upon which rests the justification [Berechtigung] for holding it to be true"** (FA, §3, p. 3).

He emphasizes that **"where there is no justification, the possibility of drawing the distinctions** [between analytic, synthetic, apriori and aposteriori] **vanishes"** (ibid.). As he adds in a note there, Frege (perhaps wrongly) believed this to be also Kant's view. Some scholars believe this was part of his deep epistemological motivation in detecting the "ultimate grounds of judgments", which was basically Kantian,<sup>11</sup> or perhaps he wanted, at this early stage, to minimize the novelty of his approach and his departure from Kant, or he may have been simply wrong about Kant here.<sup>12</sup> In any case, later on Frege was quite clear about the difference between his notions of analytic/synthetic and Kant's, to the point he could say in the conclusion of FA, §88 (in a passage to which we shall come back towards the end) that arithmetic "extends our knowledge" and is synthetic in Kant's sense of the term, though analytic in his – Frege's.

Even granting the reducibility of arithmetic to logic on the basis of some definitions (to which I shall come back shortly), for understanding the analyticity of arithmetic we still face a problem about the epistemic status of the axioms of logic:<sup>13</sup> What is their justification? In light of the above principle, they must have one if they are to be regarded analytic in Frege's sense, which very few seriously doubt.<sup>14</sup>

Some people think the question is spurious because the axioms of logic are self-evident. However, besides the intrinsic problems of the notion, and even if we would, quite loosely, regard the self-evidence of a proposition as its justification, self-evidence in itself cannot be the end of the justificatory story. For the kind of justification relevant to analyticity we need to know in what way the axioms are self-evident. To appreciate the point one should note that the axioms of Euclidean geometry are presumably self-evident (as Frege himself thought) – they were considered for centuries the paradigm of self-evidence (definitely more so than the axioms of Frege's logic). Why then isn't this sufficient for rendering them, and whatever is logically derived from them, analytic? If being self-evident would suffice for the kind of justification Frege is alluding to, (Euclidean) geometry should be deemed analytic, according to Frege, just as arithmetic

and logic. But Frege didn't think so – he thought that geometry, in spite of its axioms being self-evident, is not analytic. So, the presumed self-evidence of the axioms of logic cannot be the sole ground of their analyticity. It cannot exempt us from asking about the particular way in which they are self-evident, and the kind of justification they may have. And the root of the difference between logic and geometry here must lie in the nature of the axioms and in the different ways in which those of geometry and of logic are self-evident or justifiable.

One could perhaps retort here that Frege simply defines analyticity in terms of reducibility to logic. Hence, the analyticity of the axioms of logic, unlike those of geometry, is not in question. But surely, calling a judgment analytic because it is provable from the axioms of logic is not an arbitrary terminological decision. These axioms must be justifiable in a particular way that gives a rationale for this decision. This is enhanced by the above principle of FA. We are back then with our question: How can the axioms (i.e. each of them) be justified?<sup>15</sup>

There is an austere sense of justification, namely derivation in *Begriffsschrift*, in which they cannot. But justification, for Frege is wider than this austere sense. In several places Frege explicitly recognizes a notion of justification wider than the deductive-inferential one:

"Now the grounds which justify the recognition of a truth often reside in other truths which have already been recognized. But if there are any truths recognized by us at all, this cannot be the only form that justification takes. **There must be judgments whose justification rests on something else**, if they stand in need of justification at all. And this is where epistemology comes in" ("Logic", in *Posthumous Writings* (PW), p. 3).

This should apply also to the axioms, for there is an inherent connection in Frege between objectivity and justifiability: objective is what is justifiable, or plays a role in a justification.<sup>16</sup> Hence, in order to secure the objectivity of a domain (like geometry) its axioms do need justification, and their justification is therefore of this wider, non-deductive kind.

In "17 Key Sentences on Logic",<sup>17</sup> article 13, Frege writes:

"We justify a judgment either by going back to truths that have been recognized already or **without having recourse to other judgments**. Only the first case, inference, is the concern of Logic" (PW, 176).

So here again, much like in the previous quote, Frege unquestionably recognized this other (i.e. non-deductive) kind of justification. In these passages Frege does not yet say what this other, non-inferential way of justification is, though his talking of this other way as "epistemological" is an important hint. He did not yet have then the terminology for his notion of sense, and does not explicitly say that this other kind of justification has to do with the senses of the constituents in question. But later, in talking e.g. about geometry, equipped with his mature notion of sense, he gets much closer to explicitly expressing the connection between sense and the justification of the axioms. About (Euclidean) Geometry, it is often realized that Frege held a Kantian view, according to which geometry is synthetic a priori. What is less often realized is that for Frege (probably unlike Kant) this concerns only its justification. The syntheticity of geometry consists in the fact that its justifiability – the way the geometrical truths, basically the axioms, can be justified – requires intuition (*Anschauung*), which is at least part of the way in which geometrical objects and concepts are given to us – their senses. In a pivotal point of FA Frege considers the equivalence "The direction of  $a$  is the direction of  $b$  if and only if  $a$  is parallel to  $b$ " ( $(D(a)=D(b) \equiv a//b)$ ). He rejects reading it as defining parallelism in terms of directions. And the reason he gives is that such a definition does not respect, as it should, the ways things are given to us: "Everything geometrical must be given originally in intuition" (FA, p. 75).

Likewise, and even more to the point, he talks in a similar vein about the axioms and says that the axioms are justified on the basis of the senses of their constituents – the ways their objects and concepts are given to us:

"So long as I understand the words 'straight line', 'parallel' and 'intersects' as I do, I cannot but accept the parallels axiom... **Their sense (*Sinn*) is indissolubly bound up with the axiom of parallels**" ("Logic in Mathematics", PW 247).

Towards the end of his late article "Compound Thoughts" [*Gedankegefuege*] Frege says: "for the truth of a logical law is immediately evident from itself, i.e., **from the sense (*Sinn*) of its expression.**" And in a piece he probably wrote in the last year of his life Frege wrote: "**From the geometrical source of knowledge flow (*fliessen*) the axioms of geometry**" (PW 273). The axioms then "flow" from something; they have epistemic grounds or justification. And this, I suggest, is basically the ways the geometrical things they are about are given – the senses of their terms.

I therefore surmise that, though Frege doesn't say so in these very words, a view about this other, non-deductive form of justification can be gathered from various scattered remarks of his, and is, any way, in conformity to basic lines of his thought. A sketch of its general outline can be put as follows: The justification of axioms, in geometry, as well as in logic, is given in terms of the senses of their constituents – the ways the things they are about are given to us. These ways are different in the two cases: in geometry they consist of special (spatial) intuitions; in logic, they consist of basic features of our ability to think or reason. Calling the latter, and not the former, analytic is therefore well motivated.

### **THE ABILITY TO RECOGNIZE OBJECTS**

The above means that an epistemic and justificatory notion of sense – a mode in which something is given to us – is central to establishing the analyticity of logic, and hence – of arithmetic.<sup>18</sup> This brings us to another important Kantian strand in Frege's view, the above differences between them notwithstanding. Kant thought that recognizing an arithmetical truth requires a special intuition, *Anschauung* (hence, their synthetic nature). Frege's view, as portrayed above, should not be deemed opposed to that (though he wouldn't use the term intuition (*Anschauung*), and the nature of the intuition concerned is different): the justifiability of arithmetical (and logical) truths depends on that of the axioms – ultimately on those of logic; and this, construed in terms of the senses of their constituents, as sketched above, **depends on there being (logical) objects whose modes of presentation to us these senses are.**

Intuition is required, on Kant's view, for a representation of an object. In general, objects are recognized by concepts and intuition – the former responsible for the "unity in consciousness" of the manifold of representations; the latter – for their singularity (CPR, A103-110; cf. also B376-7). A characteristic mark of his view is that it depends on sensibility, which is Kant's general term for the mental capacity perceptive to the ways objects are given us: "Without sensibility no object would be given to us" [*Ohne sinnlichkeit wuerde uns kein Gegenstand gegeben*].<sup>19</sup> This general capacity is activated by sensations, which, as Kant makes clear, are required for what he calls "empirical intuition"<sup>20</sup>. Kant also recognizes what he calls "pure intuition" of the pure forms of intuition, namely space and time. Some interpreters find it profitable to separate these two factors, and regard intuition as required for representing objects in general, whether it

depends on sense perception and sensibility or not.<sup>21</sup> I shall not delve here on this much discussed issue in Kant.

For my concerns, the important point to note is that Frege's view of arithmetic also requires such a capacity, in spite of his rejecting what he took to be Kant's view that it depends on sensibility – whether empirical or pure. In a wide sense of "intuition" as a capacity of recognizing objects (without the restriction to sensibility) it therefore requires intuition. Frege devoted much space and effort to establishing that numbers are objects, and he regarded the question of how logical objects (in particular, numbers) are given to us as central to [the philosophy of] arithmetic.<sup>22</sup> Sure enough, a crucial thesis of Frege's is that objects (e.g. numbers) can be given us by logic and reason, independently of sensation and space and time.<sup>23</sup> And yet, as stated above, a general capacity to recognize objects and ways they are given to us is required by Frege's conception of the objectivity of logic and arithmetic. Whether this cognitive ability to recognize objects is called "intuition" or not is of lesser importance. Whether it depends on sensibility – as Kant proclaimed and Frege denied – is a more important and substantial issue. However, it should still not blind us to a main point of agreement, namely, that for both Kant and Frege, **our knowledge of arithmetic depends on our ability to recognize objects and their existence.**

When this is properly appreciated, another Kantian strand in Frege's thought emerges: the dependence of the objectivity of arithmetic on our ability to recognize objects. This is not a trivial similarity – even among logicistic approaches it is a distinctive mark of Frege's: The analyticity of arithmetic depends on that of logic, and on the justifiability of its axioms. This is accomplished in terms of the senses of their constituents. Sense, for Frege, is a sense of something – of an object or a function.<sup>24</sup> It is a mode of its being given to us. A function, in turn, depends on objects, and conceiving a function depends on our ability to recognize objects.

Moreover, it was this Kantian conviction that led Frege to insist on the existence of logical objects, which eventually led to the contradiction (the so called Russell paradox) and to what he regarded as the failure of his logicistic project. The above conviction may at least partially explain Frege's "obsession" with logical objects (truth-values and extensions of functions). For, without them it is hard to see what the senses of the constituents of the logical axioms could be senses of, and deprived of these, we have a poor notion of the justification and analyticity of the axioms of logic (and arithmetic).<sup>25</sup>



Logical axioms are conceived by Frege as universal truths, which are construed, in general as (second-order) predication on functions. A function for Frege (including concepts, which are functions to truth-values), though real and objective, is a particular way of connecting objects – connecting the arguments of the function to its value. This is its whole essence and "being". (This was emphasized and largely expanded by Dummett. See, e.g. his 1973, ch. 8.) The notion of a function therefore supervenes on that of object and talking or thinking of functions supervene on the ability to recognize objects. Throughout his career Frege maintained that functions are grasped only through their linguistic expressions. In §9 of his early *Begriffsschrift*, still lacking his sense/reference distinction and talking in terms of "contents" (Inhalt) of expressions, Frege was unable to express the crucial distinction between a content and a particular way in which it is given. He was therefore almost forced to identify a function with an incomplete expression.<sup>26</sup> Later, in his mature position, beginning with FC, he was clear that functions belong to the realm of reference and are real and objective. However, he still maintained that they are unsaturated and graspable only through the incomplete linguistic expressions that denote them. In explaining the function denoted by a functional expression like  $(2+3x^2)x$  Frege says:

"The essence of a function is revealed rather in the connection established between the numbers whose signs replace 'x', and the numbers that then appear as Bedeutungen of our expression... The expression of a function is incomplete, unsaturated. The letter 'x' merely serves as a place-holder for a numeral to complete the expression..." (BL, §1).

Though Frege uses here algebraic examples, he expanded the concept of function, in terms of both arguments and of values, to include concepts and relations, which were thus conceived as functions whose values are the two truth-values (e.g. *ibid.* §2). This is a main point in FC and has also its root in §9 of *Begriffsschrift*. Hence, concepts also have their essence and "being" in their applying to objects. Frege repeatedly emphasized the idea by saying that they are "**essentially predicative**" (e.g. CO, 182/193; Letter to Russell, 13.11.04, PMC 161).

Hence, when logical axioms are conceived as universal propositions – as predications on functions – this should not detract our claim that they are intrinsically connected to our capacity to recognize objects, because this capacity is essential for our notion of a function. The connection here is admittedly more remote than in a simple thought about a particular object, but it is still valid and important to be noticed. In talking of this intrinsic

connection I do not mean to claim for the priority of one side over the other – rather, they go hand in hand. In grasping a thought we grasp its constituent senses, which are modes in which their references are given to us. These references are either objects or functions. And when they are functions their recognition ultimately depends on that of objects. And with respect to the logical axioms in Frege's system, the functions concerned are only functions of logical objects – truth values and extensions.

One of the most obvious features of the course of argument in FA is that Frege sees it necessary to establish that "**Every individual number is a self-subsistent object**" (p. 67). He then asks the crucial and typical question: "**How then are numbers to be given to us**" (§62). This is the starting point and the pivotal move in his developing his view that numbers are logical objects that are given to us, or definable, by logic alone. He shows this first for the concept Number (§68) and then for the individual numbers (from §74). This general strategy is maintained in BL.

It has been argued (for instance, in Bar-Elli (2001), that for Frege, logical objects – the truth-values and extensions of functions – are indispensable for reasoning and thinking and are required by irresistible logical principles, like axiom V of BL (already recognized in FC) and the axioms of truth-functional logic.

In SR, after establishing the True and the False as the reference of sentences, Frege writes:

"Every declarative sentence concerned with the reference of its words is therefore to be regarded as a proper name, and its reference, if it has one, is either the True or the False. **These two objects are recognized, if only implicitly, by everybody who judges something to be true...**" (33/63)

It is also argued there that examining the justifications Frege gives to the axioms in BL (§18) suggests that the axioms of truth-functional logic are not only truths about the truth-values (or functions over them), but they express on this view aspects of the ways the True and the False are given to us as logical objects.

Frege ends (the appendix II to) BL by stating: "The prime problem of arithmetic is the question, In what way are we to conceive logical objects, in particular numbers?" (143). ("Way of conceiving" is one of the expressions Frege uses for his notion of sense, and it is virtually synonymous here with "way of being given".)

Similar remarks accompany the introduction of value-ranges in FC and BL. And in a letter to Russell of 28.07.02, even after realizing the trouble into which axiom V leads, Frege wrote":

"But the question is, How do we apprehend logical objects? And I have found no other answer to it than this, We apprehend them as extensions of concepts, or more generally, as ranges of values of functions" (PMC 140-1).

So, the recognition of logical objects amounts not only to realizing the truth of the corresponding axioms, but also to a particular way of this realizing: to a special construal of these axioms as being "about" those objects, and of their "self-evidence" as being grounded in their expressing features of the Sinne of their constituents – the ways these (logical) objects are given to us.

Hence, grasping the logical axioms, and logical truths in general, requires a capacity of recognizing objects, which in Kantian terms involves intuition (*Anschauung*). The fact that in many cases this is a very special kind of intuition, which constitutes our ability to think and to reason, is of course important and marks the location of a possible difference between them,<sup>27</sup> but it should not blur the substantial common ground: The notion of objectivity and our ability to have objective judgments – arithmetical ones included – depend on there being objects given to us in particular ways. This then is another Kantian strand that interweaves into the Fregean view of the basis of the justification of arithmetic – hence of its analytic nature. And note – "analytic" is here in Frege's sense.

### **BEAMS AND SEEDS - FRUITFUL ANALYTIC DEFINITIONS**

But the point has also to do with the nature of definitions and their role: for Frege, analytic, let us remember, is what is justifiable by (or reducible to) logic and definitions alone (FA, §3). These definitions (of the basic arithmetical terms) cannot be philosophically unconstrained. If they were, any consistent (first order) theory would be analytic, for any such theory is reducible to logic by some set of definitions. The definitions of Frege's reduction of arithmetic to logic, moreover, are not only what he calls "constructive" or stipulative definitions of new terms. Rather, they are what he calls "analytic" (*zerlegende*) definitions of terms in use, whose meanings are partially and perhaps dimly recognized.<sup>28</sup> Hence, these definitions must satisfy some constraints.

What constraints? This brings in a wide and complicated subject, on which I cannot dwell here.<sup>29</sup> Let me just hint at one point. Geometry (Euclidean geometry) has a model in arithmetic. That is, there are "definitions" of the geometrical terms, by which the axioms of geometry would be truths of arithmetic. Hence, if definitions would be left unconstrained, since arithmetic is reducible to logic so would Geometry, which should then be deemed analytic (in Frege's sense). All this was of course well known to Frege,

and yet he rejected the conclusion. The way Frege would have blocked this move is, I guess, by philosophical constraints that should be imposed on the "definitions" of the geometrical terms: such modeling of geometry in arithmetic would not satisfy these constraints. This idea is important for understanding a central move in the course of the argument of FA, to which we have already alluded, where a certain provisional way of defining expressions of the type "the number of Fs" is rejected. In explaining the rejection Frege appeals at a certain stage to a geometrical example in which the equivalence  $D(a)=D(b)\leftrightarrow a//b$  [the direction of a is the direction of b iff a is parallel to b] is taken as a definition of parallelism. Frege rejects this because **"this is to reverse the true order of things"** [*der wahre Sachverhalt damit auf den Kopf gestellt wird*] **"For everything geometrical must be given originally in intuition"**. Parallelism is given in intuition, Frege thought; directions are not (FA, §64, p. 75). Therefore one cannot define the former in terms of the latter. The general lesson to be learned – a lesson without which FA and in fact his whole project, cannot be understood – is that the pertinent definitions are philosophically constrained, where a dominant consideration here has to do with the ways the things concerned are given to us.

As stated above, I cannot delve here on the nature of the required constraints on analytic definitions. What I would like to suggest here is that the fact that the pertinent definitions are "analytic" – are of terms in use whose meaning is partially grasped – introduces another Kantian strand into the picture. To put it roughly, an analytic definition explicates the meaning of a term in use, and the conclusions logically drawn from such definitions are, in some sense, contained in the definitions, hence – in the meanings of these terms. Let us see a crucial passage here: Towards the end of FA, after explaining that the poverty of the logical structure of traditional (and Kantian) definitions is artificially restrictive in using the boundary lines marked by the old concepts (of the *definiens*), Frege continues about his own definitions:

"But the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all [...] **The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic**; and yet they can be proved by pure logical means, and are thus analytic. The truth is that **they are contained in the definitions** (*in den Definitionen enthalten*), but as plants are contained in their seeds, not as beams are contained in a house. Often we need several definitions for the proof of a

single proposition, which consequently is not contained in any one of them alone, yet does follow purely logically from all of them together" (§88, p.100-1).

This passage is particularly rich and illuminating. Frege in fact makes here clear that the issue between him and Kant is not so much about the nature of arithmetic as about the notions of analytic and synthetic, and about the nature of definitions. Frege claims, in agreement with Kant, that arithmetic, i.e. the conclusions logically drawn from the definitions concerned, extends our knowledge, and is synthetic in Kant's sense of the term.<sup>30</sup> His claim that it is yet analytic displays his awareness that they were using these terms differently. And yet, Frege expresses here also his sophisticated version of Kant's famous "containment condition" of analyticity: Arithmetical truths are contained in their concepts (or their definitions) as plants are contained in their seeds. Thus, Frege subtly trades here on two central Kantian themes: on the one hand, arithmetic extends our knowledge (and is, in Kantian terms, synthetic), while on the other, it is in some sense contained in the meanings of its terms (and is thus analytic).

It might seem that Frege's "containment", which is a relation between judgments, is very different from Kant's, which is a relation between concepts. But this doesn't seem to me to go very deeply, for the containment of e.g. 'round' in 'circle' is no different than the containment of 'x is round' in 'x is a circle'. And when Frege speaks of analytic truths as derivable from definitions alone he sometimes speaks of them as being derived from a concept. About the laws of identity, for instance, he says: "**As analytic truths they should be capable of being derived from the concept alone** [*aus dem Begriffe selbst*]" (FA §65, p.76).<sup>31</sup> Moreover, Kant's notion is not strictly a relation between concepts, for he thought of syllogisms like "If all humans are mortal and all Greeks are human, then all Greeks are mortal" as analytic.<sup>32</sup> It is plausible to assume that he would regard the conclusion of a deductive inference, at least in simple cases like such syllogisms, as "contained" in its premises. Hence, the difference between construing containment as a relation between concepts or judgments is not crucial here.

A definition in itself, according to Frege, is not a statement that can be analytic or synthetic.<sup>33</sup> In the above passage from the conclusion of FA Frege speaks of the fruitfulness of (his) definitions – a topic which recurs in his writings. The notion of fruitful (*fruchtbar*) concepts (definitions) and its metaphorical presentation as setting new borderlines (carving new areas of reality) is central already in the early "Boole's Calculus Logic" (BLC) (PW, see especially pp. 33-35).

It is not easy to be precise on what Frege meant here, but some points seem clear. As Frege repeatedly claimed the definition must be operative in proofs of significant theorems in the field. This is a sign of the definition's analyzing the content "at its real joints", where the structure and order of dependencies in the field concerned, and connections between propositions and concepts within it are manifest. As Frege says: "The insight it [a genuine definition] permits into the logical structure... is a condition for insight into the logical linkage of truths" (CP 302). A characteristic Fregean facet of such analyses is that they are achieved in terms of (nested) quantification and variables. The Weierstrass definition of the continuity of functions served as paradigmatic example:  $F(x)$  is continuous at  $a$  iff  $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)((x-a) < \delta \rightarrow (f(x)-f(a)) < \epsilon)$ .

The fruitfulness concerned is another expression for the extending knowledge of which Frege speaks in the above passage. This, as he says there, is usually a matter of deriving a proposition from several premises and definitions. But it may also be a derivation from one sentence, if the proof of the conclusion is not trivial and immediate.<sup>34</sup> In any case, as e.g. Tappenden (ibid.) emphasizes, the extension of knowledge concerned is not merely psychological: it is objective and concerns the objective order, connections and structure of the (mathematical) field.

We said before that strictly, a definition, on Frege's view is not a statement. There are, however, two points to note here in which a definition may give rise to a significant statement: (1) The definiendum, the term defined, may be a significant term in use, whose meaning cannot be disregarded. The definition in that case is very close to a statement analyzing this meaning or use in a way that must conform to it – what Frege calls (in "Logic in Mathematics", ibid.) explication (*Erleuterung*).<sup>35</sup> (2) The definiens may, and in most interesting cases would, express or catch a "new" concept – "new" in the sense that, relative to its constituents, it carves up reality (the content) in a new way.

In BLC and FA Frege emphasizes that a definition is not just a conjunctive or disjunctive combination of characteristic properties (*Merkmale*), but that **"every element in the definition is intimately, I might almost say organically, connected with all the rest"** (FA, §88, p. 100). Evidently, Frege conceived of this "organic connection" in terms of logical structure and in particular – quantification and bound variables.

Suppose, for example, you have two concepts A and B. You can define on their basis a concept " $Cx \equiv (Ax \ \& \ Bx)$ ", which would apply to anything that is both A and B. But this, Frege claims, would be trivial definition, not forming a really new concept, because

it uses and is restricted to the "boundary lines" (if we present concepts by bounded areas) of the old concepts, A and B. On its basis, statements like "All Cs are As" are (trivially) analytic both in Kant's and in Frege's sense. But one may also define a new concept that, by its logical structure, would "delineate a new area in reality", not confined to the boundary lines of the old concepts in terms of which it is defined. The same point, with the same metaphor of drawing boundary lines, is central in Frege's early BLC, and it applies not only to conjunction but to any combination of Boolean operators. Frege protested against Kant (and Boolean logicians) that he thought only of definitions of the first kind, which is a severe and artificial restriction, while the interesting definitions in logic and mathematics (including his own in *Begriffsschrift* and FA) are of the second kind. Take for example Frege's definition of *successor* in a series: The definition is couched in terms of the notion of hereditary property, which, in turn, is defined in terms of the general notions of a property and of a relation or function: G is hereditary in a series defined by f iff any object, which bears f to an object which is G, is also G [ $H(G,f) \equiv (x)(Gx \ \& \ f(x,y) \rightarrow Gy)$ ]. Using this, Frege defines successor thus: y succeeds x in a series f, iff y has any hereditary property in the series that any object bearing f to x has [ $S(y,x) \equiv (z)(G)((H(G,f)\&f(z,x) \rightarrow Gz) \rightarrow Gy)$ ].<sup>36</sup>

When specific properties and relations are concerned, one can see what Frege means by saying that the definition of successor "carves a new area in reality". Suppose, for example, you have two properties – wise and tall – and a relation – being the son of-. Suppose moreover that being tall is hereditary with respect to the relation being son of-. One can then define successor as above, and the idea of successor would be "new" with respect to this modest basis; it would "draw a new boundary line in reality". There is a big difference between the boundary lines of the properties in the base and the one set by the definition. And this difference is traversed by logic – by the logical structure of the definition – a structure which also exhibits and gives us an insight into implication relations between propositions, "the logical linkage of truths" (CP, 302). Frege can then prove results like: "whatever bears f to a successor of x is itself a successor of x", which are "new" and extends our knowledge.<sup>37</sup> Hence, they are synthetic in Kant's sense (and yet analytic in Frege's). This then is one element in the "containment" Frege speaks about, by which the other Kantian strand I mentioned emerges: the result is contained in the defined concept as plants are contained in their seeds – logic and the logical structure of definitions is assimilated here to the biological process of growth. And as natural it is

for us to say that a plant is "new" relative to its seed, so, given the biological process of growth, it is to say it is still "contained" in its seed.

But there is another element in Frege's talk of "containment". A definition, as noted above, may be "analytic", i.e. of a term in use whose meaning is grasped (if only dimly and partially) independently of the definition, as is the case in most Fregean definitions of arithmetical terms. Being "analytic", what is expressed by the definition, even if it draws new boundary lines, is in some way "contained" in the sense of the term defined. The above definition, for example, not only shows that the idea of successor is in some sense contained in that of property and relation (as plants in a seed), but being an analytic definition it also must display the defined notion of successor and conclusions drawn from it as being "contained" in the pre-theoretic idea of succession. This should not be seen as casting any doubt on Frege's valid point about the fruitfulness of his definitions and about their carving new areas in reality. But "new" here is in respect to the old concepts in terms of which the definition is couched. This doesn't conflict with the claim made here that this "new" area in reality is already "contained" (in the sophisticated Fregean version) in the sense of the defined term as used independently of the definition. Sure, a Fregean definition (or explication) goes beyond and deeper than its pre-theoretic "origin", but it is not entirely new relative to it: it is, after all, a definition and explication of it. Hence, there is no absurdity in a sort of a Kantian claim that even with regard to such Fregean fruitful definitions, say that of successor, whatever is proved by means of it is already, in some way, contained in the sense of "successor".

"In some way" is here illusive – there is an important difference between the way Kant thought about this containment, and the way it can be modified to apply to Frege. First, as Frege himself put it in the above quotation, there is the containment of beams in a house, and that of plants in their seeds. Kant's was the first; his (Frege's) – the second. Even when restricted to stipulative "constructive" definitions, Frege's notion is much richer and logically more sophisticated. It also gets cleared of whatever psychological overtones one may hear in the Kantian notion (in terms of what one actually thinks in grasping such a truth). Secondly, there is a great difference between the containment in a Kantian definition, and that in a Fregean explication. And yet, it is significant that Frege found it appropriate to stick to the ideas of analysis and containment here. Deeply – this is also a Kantian strand.



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## Notes

- <sup>1</sup> I focus here on *Die Grundlagen der Arithmetik* (Breslau, 1884) – *The Foundations of Arithmetic*, translated by J. Austin, Oxford, 1950 (henceforth abbreviated as FA). One should bear in mind that Frege was clear then that he still hadn't shown this conclusively, and was aware of possible doubts (FA, §90).
- <sup>2</sup> This is a widespread, almost standard, conception: For a recent instance, see S. Yablo (2008) where he characterizes Frege's notion of analyticity as unformativeness (p. 154). This in Yablo is a casual aside, which doesn't bear on the important content of his paper, but just for that it reflects a widespread conception.
- <sup>3</sup> "Kantian strands", because some of the pertinent Kantian ideas are unclear, and though as I shall argue, they play a role in Frege's view, this does not mean that he endorsed them as they are. These three strands go beyond Frege's general applause to Kant's raising the analytic/synthetic distinction with respect to mathematics – reflecting, as it does, a search for the "ultimate sources of our knowledge" – and for his doctrine of the synthetic a priori (FA, §89).
- <sup>4</sup> I shall leave geometry aside here. Let me just mention that Frege's admiration to Kant inclined him to undermine their differences (cf. FA, § 89 second paragraph). This bears also on their use of "intuition" (*Anschauung*), which Frege, unlike Kant, applied also to concepts and functions. I shall also ignore his possible change of view after the "Russell paradox", and particularly at the last year of his life.
- <sup>5</sup> See the Introduction to the *Critique of Pure Reason*, translated by N. Kemp Smith, Macmillan, 1933, B11. There are several other formulations in Kant, in terms of the "law of identity" and the "law of contradiction". Most commentators agree that they all concern the content of a judgment, though some have challenged it; see, for instance, van Cleve (1999).
- <sup>6</sup> See, for instance, MacFarlane (2002).
- <sup>7</sup> I leave "Kantian strands" rather imprecise, for, as stated in the text, my main concern is with Frege's views, not Kant's, and I intentionally avoid entering into scholarly debates about Kant's exact view.
- <sup>8</sup> See his SR. The basic idea is central already in §8 of his 1879 *Begriffsschrift*, (1967); for a detailed discussion see Bar-Elli (2006). It is operative again in FA, e.g. §67. This is the primary characterization of sense in SR. Later, in his BL, Frege talked of sense mainly as a constituent of thought. For a discussion of the relationships between the two see Bar-Elli

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(2001). Another characterization, which Dummett made prominent – as a "way of determining the reference", or a "route to the reference" – seems to me unhappy and has a slim basis in Frege; and a widespread explication of it, as a condition whose sole satisfier is the reference, seems to me strictly wrong in suggesting a predicative view of sense in which it is a concept or a property, which is opposed to Frege's explicit view.

<sup>9</sup> Though some nuances in the above formulations are my own, the basic ideas have been much discussed at least since Dummett's (1973). See for instance, Carl (1994); Bar-Elli, (1996).

<sup>10</sup> This has been argued in detail in Bar-Elli (2010), 165-184, and in Bar-Elli 2009

<sup>11</sup> See for instance, Weiner (1990), 54-55.

<sup>12</sup> If Frege was right about Kant here (as some Kant scholars believe) this is another Kantian strand in his view, and it should not make much change to the main claims here.

<sup>13</sup> In FA § 17 Frege considers a possibility that arithmetic is proved as a set of analytic conditionals whose antecedents are the logical axioms, just like geometry, with its axioms as antecedents. In both cases the conditionals themselves are analytic, i.e. proved by the logical axioms, which are supposed to be the antecedents in the first case.

<sup>14</sup> Dummett, and many others, have taken the analyticity of logic and its axioms as almost a truism. Dummett accused Frege of carelessness in formulating his definition of analytic for that reason (See his (1991), p. 24). Burge, for one, claimed that Frege followed Kant in not regarding the axioms of logic to be analytic (Postscript in his (2005), p. 388). I can't get here into a detailed discussion of this point, and shall just state that in my mind there are good reasons to think (with Dummett and against Burge) that Frege did and should have regarded logic (and its axioms) to be analytic. However, what Dummett considered carelessness in Frege's formulation is due to his failing to see the justificatory nature of Frege's notion of analytic (the above quoted principle) and the fundamental role of the notion of sense in justifying the axioms (though not deductively). Dummett holds both that logical axioms are analytic and axioms of Geometry – a priori, and that the only form of justification is deductive inference, and axioms cannot be thus justified. But these two are incoherent with Frege's explicit principle (quoted above) that the notions of analytic, a priori, etc. concern the justification of a proposition. I have expanded on this in Bar-Elli 2009 and 2010 *ibid*.

<sup>15</sup> According to an influential trend in Frege's scholarship, the principles of logic cannot be justified. This is often connected with the view that there is no "Meta-logical perspective" to use Ricketts phrase (See also Weiner (1990). However, first, one can

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concede the latter without the former: justifying logical principles may be conceived as an inner logical enterprise. Secondly, the unjustifiability of logical principles is at flat contrast with their analyticity, given Frege's justifiability notion of analyticity (Bar-Elli (2010). Thirdly, besides some of Frege's formulations with which such a view is at odds, he does provide (verbal) justification for his logical axioms (Burge, 1998). Weiner (ibid) regards all these as "elucidations", which, she emphasizes, are not proofs in the system. She is of course right in that but seems to ignore the claim at issue, namely that proof is not the only form of justification – there are justifications, which are not (deductive) proofs in Frege.

<sup>16</sup> This is argued in detail Bar-Elli (2010).

<sup>17</sup> The editors of the *Nachlass* date it not later than 1906. Dummett suggested it is a much earlier piece, see his *Frege and Other Philosophers*, Oxford, 1991, pp. 66, 77.

<sup>18</sup> This option is absent in the discussion of Frege's views on the justification of basic logical laws in Weiner (1990), ch. 2. Moreover, Weiner is unclear on whether primitive logical laws are justifiable: On the one hand, she recognizes that "not all justification, on Frege's view, is inferential" (p. 61), and "the justification of a primitive logical law is evident from its content" (though not by the mental act of considering it, p. 78), while on the other she also says that "primitive logical laws cannot be justified" (p. 77).

<sup>19</sup> Kant, I.: *Critique of Pure Reason*, 1933, A51/B75.

<sup>20</sup> See for example, Kant, op. cit. A 19.

<sup>21</sup> See, for instance, Hintikka, "An Analysis of Analyticity", in his (1973), 123-149, p. 145, and other articles referred to there.

<sup>22</sup> Recognizing numbers as objects is a central thesis of FA and vital for its course of argument. Frege repeatedly presents the question about how they are given to us as pivotal to it. He begins §62, for instance, by asking "**How then are numbers to be given to us?**", and answers that since numbers are objects, a clue to the answer is to define the sense of numerical identities. And at the very end of the appendix (*Nachwort*) to *Grundgesetze* (BL) he refers to it again as the key question: "**The prime problem of arithmetic is the question, In what way are we to conceive logical objects [wie fassen wir logische Gegenstaenden], in particular, numbers? By what means are we justified in recognizing numbers as objects?**" (BL, 143).

<sup>23</sup> The point is clearly made e.g. in FA §89.

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- <sup>24</sup> This is a notoriously debated point. Some of Frege's formulations suggest that there are referenceless senses. But this, I believe, is not his better and considered view. His principal characterization of the sense of a term is a mode in which its reference is given to us, and it is hard to make sense of this where there is no reference. Moreover, a thought, which is the sense of a proposition, and is built up by the senses of its constituents, is essentially true or false, which according to Frege is impossible when some of its terms lack reference. And indeed, in his logical language, Frege takes care to ascertain that all terms have references, and repeatedly claims that this must be the case in any "scientific" language. Examples he gives from natural language of names lacking reference (e.g. Odysseus) are of fictional characters in artistic, non-scientific, contexts, and he was an extreme subjectivist about art, claiming that it does not express genuine thoughts, but "apparent thoughts" (*Scheingedanken*, sometimes translated "mock-thoughts"), and he sometimes speaks of such names as "apparent names" (*Scheinnamen*); Cf. Bar-Elli (1996), chapter 3.
- <sup>25</sup> This has been argued in detail in Bar-Elli (2001).
- <sup>26</sup> See on this Bar-Elli (2006), where some alternative views are examined.
- <sup>27</sup> It is arguable that in Kant intuition is also a pre-requisite for our ability to think, but the meaning and the argument for this are different than in Frege, where it is much more straightforward. In any case, such a reading of Kant would strengthen this Kantian strand in Frege.
- <sup>28</sup> See "Logic in Mathematics", in his PW, 227/210-211. Some authors ignore this important notion. Weiner (1990, ch. 3) for instance not only ignores it but adopts such an austere and restricted notion of analysis that makes it incoherent, which shows that it can't be Frege's notion.
- <sup>29</sup> This is expanded in Bar-Elli 2009, mainly chapter 7.
- <sup>30</sup> The basic idea is repeated at the end of § 91, where he says that "**propositions which extend our knowledge can contain analytic judgments**".
- <sup>31</sup> This, by the way, in contrast to Burge's view that Frege did not regard logical principles to be analytic (see note 14 above). There are other reasons to doubt Burge's position here, e.g. that Frege was aware that the same proposition can be an axiom in one system, while provable in another, logically equivalent one, but it is hard to adopt this kind of relativity to the notion of analyticity. I shall not go into the details here and just assume (with Dummett and many others) that Frege would regard logical axioms to be analytic, and geometrical axioms to be a priori.
- <sup>32</sup> See C.D. Broad (1978), p. 4.

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- <sup>33</sup> Cf. *Begriffsschrift*, §24; FA §67, p.78; PW p. 224-6/207-9.
- <sup>34</sup> See on this Tappenden (1995); on this particular point see p. 435, where he ascribes the idea to Dummett's (1981), p. 300.
- <sup>35</sup> Cf. Kambartel's introductory chapter to Frege's *Nachgelassene Schriften*, XVII-XXV.
- <sup>36</sup> Following current usage I use "successor" for Frege's "follower" [*Folgenden*], in distinction to "immediate successor" [*Naechstfolgenden*].
- <sup>37</sup> Cf. Frege's slightly more complicated example at the end of FA §91.

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