Sense and Analysis – Studies in Frege

Gilead Bar-Elli

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Contents:

Introduction

The book is based on articles that were published in philosophical journals and books over a span of many years. Except for small editorial corrections, I have tried to minimize changes in the text of the articles. Though the articles were independently written and are devoted to various topics in Frege's philosophy, the book is still informed by some basic strands, which make it, I hope, coherent. Chief among them is an interpretation I propose for various aspects and implications of Frege's notion of sense (*Sinn*), which at some crucial points is different from current common ones.

Frege's notion of sense and the distinction between sense and reference are often presented minimally as his special, almost ad hoc proposal to explain the cognitive value of identity statements, roughly in the following way. Frege's famous "identity puzzle" asks how an identity like "The morning star is the evening star" can be of cognitive value, since, if true, it says the same as the trivial "The morning star is the morning star". His answer is that in the first case the two sides of the identity have different senses (while their references are the same). Somewhat more charitably, and with a better and deeper understanding, his notion of sense is regarded as a general distinctive aspect of his theory of meaning. Roughly, meaning, on that view, has two "dimensions", one is reference (*Bedeutung*), which is what statements and thoughts are about and what determines their truth and falsity; the other is sense (*Sinn*), which concerns a mode of presentation of the reference, or how it is conceived, as this is expressed by terms referring to it. This evidently is a substantial, wide and deep philosophical theory that in fact started modern theories of meaning.

But even this falls short of realizing the full significance of the notion of sense in logic, ontology, epistemology, philosophy of mind and other philosophical concerns. Frege's logic has been generally adopted and forms the basis of standard modern logic, and the essentials of his theory of reference form the basis of elementary formal semantics. His notion of sense, however, is seldom taken seriously in modern logic, and some proposals to explicate it in formal terms – mainly in terms of possible worlds in modal logics – sin to essential elements of his notion. There are, of course, many allusions and remarks in the literature concerning the notion of sense in other areas of philosophy, but they are often casual, sometimes negative, and almost always made from a limited perspective of a specific issue (like e.g. propositional and de se attitudes). However, Frege's notion of sense is, I believe, one of the profoundest and most general

ideas in philosophy, whose significance touches almost any area or topic. Sure, Frege's notion of sense cannot be detached from his theory of reference; In fact I shall go further and argue that sense is supervenient on reference, and this is vital for understanding its role in a theory of intentionality (and makes Frege also the founder of "externalism"). But it has wider significance that concerns many other fundamental philosophical notions, on which Frege was much sparser in his explicit pronouncements. Evidently, on all these other philosophers have other views and other approaches, which may have their own merits, but his way, I believe, has often not been given its due weight.

The prime significance of the notion of sense is in the philosophy of mind and a theory of intentionality. Thoughts, in Frege's theory, are senses (of complete sentences) whose constituents are senses (of sub-sentential terms). Whenever we think or entertain a thought about things in the world we do it by grasping senses. Hence, belief, knowledge and other so called propositional attitudes, which comprise many of our mental concepts, are concerned with senses. These would make the notion important enough. But other basic notions like truth, objectivity, justification, intentionality, proof, analysis, complexity, causality, explanation, etc. turn out to involve or even depend on the notion of sense. Some will be dealt with in subsequent chapters. Like with many other ideas of Frege's, his notion of sense, as a way in which something is conceived as this is expressed by the linguistic terms referring to it, has been absorbed in the philosophical world and influenced it to such an extent that it is often hardly noticed. Some examples will be discussed in the following chapters. To give one more example, which I don't discuss in this book, I would mention the prevalent use of "under description" idioms (like "event under description"), made popular in wide circles since the 60s of the last century by Morgenbesser, Davidson, Kim and others. Putting aside some unclarities and problems in the use of these expressions, it seems that this is a special case of the general Fregan doctrine of conceiving a reference "under a sense".

I put special emphasis on Frege's characterizing sense as a mode of presentation or of being given or of conceiving a reference, expressed by linguistic phrases referring to it (I call it the "core idea" of sense). Genuine senses are thus supervenient on their references. Regarding the use of indexicals and demonstratives (like "I", "here", "now", "this", etc) Frege remarks that elements of the contexts are also parts of the senses concerned, which enhances the above supervenience on reference and the externalistic

predilections. I present this as distinct from other characterizations, like e.g. Dummett's influential "route to the reference" idea. This, I believe, has a slim basis in Frege, and a common explication of it as a condition whose sole satisfier is the reference seems flatly wrong in suggesting a predicative construal of the relationship of sense to reference, to which Frege explicitly opposed. Consequently I play down the alleged role of sense in the use of terms lacking reference and in sentences that are neither true nor false. These, evidently, need explanation, but focusing on these (as unfortunately is quite current in many interpretations) is not only unfair to Frege's main lines of thought, and conflicts with many of his pronouncements, but also distracts from its main significance, and misleads as to its ontological character, its epistemic nature, its explanatory and justificatory role and its crucial significance in a theory of intentionality and the capacity of thoughts to be about things in the world. Some alternatives (including perhaps Dummett's) are also philosophically problematic and make the relationship between sense and "its" reference (when it has one) quite mysterious. Many of these problems don't arise, or are satisfactorily met, when the core idea of sense, as mentioned above and elaborated in the book, is properly understood.

On my interpretation of the notion of sense it is the basis of Frege's positions on, and contributions to many perennial philosophical problems in ontology, epistemology, logic and the philosophy of language. Let me briefly mention some:

1. Sense, as is well known, is crucial in Frege's theory of meaning and in his view (still widely rejected) that such a theory must be "bi-dimensional" (reference and sense), but its significance – particularly in seeing sense as supervenient on reference – and his reasons for this, as well as its implications, are, in my mind, still not fully appreciated.
2. Since logic is conceived by Frege as a meaningful language consisting of universal truths (as against a formal calculus to be interpreted in various models) the above is crucial for his conception of logic.
3. Senses have profound justificatory function, which is crucial particularly in justifying basic truths (axioms) of a domain. Consequently, it is pivotal also in Frege's notions of analyticity and apriority, which concern and are determined by justifications. This is another aspect of the centrality of sense for his conception of logic and his logicistic project.

4. Senses are the objects of philosophical (including logical) analysis, and the principles governing his notion of sense are the main constraints on the adequacy of analysis.
5. Senses and conceiving a thing under a sense are crucial for the notions of structure and complexity. Applying these notions to things (references) disregarding their senses is incoherent.
7. Senses, though real and objective are neither objects nor functions, though supervenient on them, thus forming a different ontological category per se, which cuts across traditional dichotomies (like the physical/mental one) and forms a basis for a profound conception of intentionality.
8. These, and some other features of sense, determine important links between Frege's philosophy and important strands in the history of philosophy, particularly Kant's.

I have deliberately tried to avoid repeating discussions that were elaborated in my previous *The Sense of Reference - Intentionality in Frege* (W. De Gruyter, 1996), even when these seem pertinent to a comprehensive understanding of the topic, and at various points have referred the reader to that book.

Though I inserted some corrections and additions to the original articles I have tried to keep them to the minimum. The book is based on the following previously published articles:

Chapter 2 on "**Frege`s Early Conception of Logic**", *Epistemologia* VIII, 1985, pp. 125-140.

Chapter 3 on **"Sense and Objectivity in Frege's Logic**", Newen et. al. (edits*). Building on Frege*, CSLI, Stanford, 2001, pp. 91-111.

Chapter 4 on "**Identity in Frege's Begriffsschrift**", *Canadian Journal of Philosophy*, vol. 36/3, 2006, 355-370.

Chapter 6 on "**The Ontological Status of Senses (*Sinne*) in Frege**", in *Metaphysics: Historical Perspectives and its Actors*, Ricardo Barroso Batista (Ed.), DOI10.17990/RPF/2015 71 2 0000.

Chapter 7 on "**Analyticity and Justification in Frege**", *Erkenntnis*: Volume 73, Issue 2 (2010), page 165-184.

Chapter 8 on "**Three Kantian Strands in Frege's View of Arithmetic**", *Journal of the History of Analytic Philosophy*, vol 2 no. 7, 2014, 1-21.

Chapter 9 on "**Conceptual Analysis and Analytical Definition in Frege",** forthcoming in *European Journal of Philosophy*

Chapter 10 on "**A Fregean Look at Kripke's Modal Notion of Meaning**", in: *Naming, Necessity, and More: Explorations in the Philosophical Work of Saul Kripke*, ed. J. Berg, Palgrave McMillan, 2014).

Chapters 1 and 5 use materials previously published in Hebrew.

I wish to thank editors and publishers for permitting to use materials in these sources.

Synopsis

Chapter 1 is introductory to Frege's logic. Though it is self contained and is designed to be comprehensible to any serious reader, it aims mainly at those who know elementary logic, as it is commonly taught in many philosophy departments today. Following a brief historical survey of Aristotelian and Boolean logic, some of Frege's main innovations, definitions and results, including the basis of his logical reduction of arithmetic, as well as an introduction to his special notation are presented.

Chapter 2 is devoted to more philosophical aspects of Frege's conception of logic. I argue that besides his epoch-making revolution and technical achievements in logic, Frege had a profound philosophical conception of logic and its nature. This is revealed in the presentation and defense of his own system in *Begriffsschrift* and in his critique of other systems. At the center of his conception is the idea that logic is not a *calculus ratiocinator* – a system of techniques for proving and solving problems – but a *lingua characterica*, combining a theory of meaning – showing how the meanings of propositions are constructed out of the meanings of their components – and a theory of inference – proving and explaining implication relations between propositions. The special novelty and power of this conception is that these are conceived as two aspects of the same theory. It is marked by its scope and "homogeneity" – that the same notions and operations function in both aspects throughout the entire language.

Central to all subsequent chapters is the notion of sense (*Sinn*). It may seem that Frege’s notion of sense, as mode of presentation of reference, has no role in his Logic (even granting its role in accounting for other aspects of language and thought). As against this I argue in chapter 3 that for understanding this role one needs to understand "logic" in its wide Fregean sense as the science of objectivity and justification. It is then argued that sense, as a mode of presentation of objects (and other entities), is vital for accounting for the objectivity of the basic truths of a domain, logic itself included. These basic truths are justified by expressing features of the modes of presentation of the objects of the relevant domain. This basic idea is exemplified in examining the way logical objects are introduced by fundamental logical principles, which express features of the ways these objects are given to us. Sense was also operative in Frege's account of implication relations of sentences in "oblique contexts" and of their meaning.

According to a widespread reading, in *Begriffsschrift* §8 Frege presented a "meta-linguistic" construal of identity, according to which identity is a relation between signs, and statements of identity are about signs; in *Ueber Sinn und Bedeutung*, according to this reading, Frege criticized and rejected it.
I argue in chapter 4 that both claims are wrong: The main argument for rejecting the meta-linguistic construal is stated already in *Begriffsschrift* §8, and the main point there is that for a coherent account of the meaning of identity statements we must consider both a content and a way of determining it. It is also proposed that Frege's implicit terminological distinction there between signs (*Zeichen*) and names (*Namen*) indicates this, where a sign just signifies a content, while the meaning of a name is a content with a way of determining it. In *Ueber Sinn und Bedeutung* Frege did not reject but endorsed this conception, systematizing it in terms of the distinction between reference and sense (instead of content and ways of determining it) and generalizing it to all expressions in any context. Thus, a strong case for a "thick" "bi-dimensional" semantics was established. Some further issues are examined in light of this interpretation.

In chapter 5 I discuss some aspects of a Fregean attitude to Russell's celebrated theory of descriptions, and why he didn't endorse Russell's theory. I argue that the answer hinges on basic issues concerning the notion of meaning and the relationships between logical structure and intentionality – what the propositions concerned are about. Frege had his own "description function". However, applying it to natural language raises some problems. Russell's theory may not face these problems, but his basic notions of meaning – proposition, denotation, incomplete symbol etc. – seem confused or incoherent from a Fregean point of view. Frege's conception of logical structure as determining both the meaning of sentences (on the basis of their constituents) and their implication relations is informed by the notion of about – what a proposition is about. Based on all this, Frege insisted on the intuitive idea that a descriptive statement is a singular statement about the object referred to by a proper name. Hence he might have thought that Russell's theory, which construes such statements as general existentially quantified ones, was flawed. I conclude with suggesting a re-interpretation of Fussell's theory, which may blunt the sting of these ciriticisms.

Chapter 6 is devoted to the ontological status of senses. I argue that Fregean senses are real and objective, but are neither objects nor functions. They are real because of their

objectivity and their being references in oblique contexts. And yet they are not objects: they don't have the mode of being of objects – independent self -subsistent identifiable entities – neither are they functions. Thus, Frege's ontology includes another ontological category, that of sense, having its own special mode of being. Among its further characteristics are its intentionality, its supervenience on reference, its being graspable by the human mind and constrained by language. Thus, it crosses simple ontological distinctions like the mental/material one, and the quite common description of it as "Platonist" is misleading.

Chapter 7 discusses Frege's notion of analyticity, and its difference from more common ones. A basic Fregean principle, governing his notions of analytic and a-priori, implies that justification, even in logic and mathematics, is a wider notion than deductive proof. In this wider sense the axioms of logic, which are analytic, and those of e.g. geometry, which are a-priori, are justifiable though not provable. Part of the importance of Frege's notions resides in this justification pertaining to *Sinne* – modes in which things the axioms are about are given to us – thus linking propositional knowledge to cognitive relations to things. According to the standard conception analyticity suggests an answer to the problem of apriority by splitting truth as between "true by facts" and "true by meanings". This split is foreign to Frege, whose answer, in contrast, respects the homogeneity of the notion of truth, which is another facet of its significance.

On the background of explaining their different notions of analyticity, their different views on definitions, and some aspects of Frege's notion of sense, three important Kantian strands that interweave into Frege's view are exposed in chapter 8. First, Frege's remarkable view that arithmetic, though analytic, contains truths that "extend our knowledge", and by Kant's use of the term, should be regarded synthetic. Second, that our arithmetical (and logical) knowledge depends on a sort of a capacity to recognize and identify objects, which are given us in particular ways, constituting their senses (*Sinne*). Third, Frege's view of definitions and explications, which in a way gives new substance to Kant's leading idea of analyticity, namely, the containment of a truth or a concept in another. In all these, Frege's view does not endorse the Kantian strands as they are, but gives them special and sometimes quite sophisticated twists.

Chapter 9 is devoted to Frege's notion of analysis – one of the central notions in his philosophy and in analytical philosophy in general. I argue that logical (or conceptual) analysis is, in Frege, primarily not an analysis of a concept but of its sense. Five Fregean philosophical principles are presented as constituting a framework for a theory of logical or "conceptual" analysis, which I call analytical explication. These principles, scattered and sometime latent in his writings, are operative in Frege's critique of other views and in his constructive development of his own view. The proposed conception of analytical explication is partially rooted in Frege's notion of analytical definition. It may also be the basis of what is required of a reduction of one domain to another, if it is to have the philosophical significance many reductions allegedly have.

In chapter 10 Kripke's seminal *Naming and Necessity* is discussed from a Fregean point of view. It is argued that Kripke's work can be read as launching an attack on a cognitive conception of meaning, and not just on a descriptive theory of the meaning of names, which Kripke ascribes to Frege. I propose that Kripke propounds a novel modal conception of meaning, which is not opposed to Frege's cognitive one. I critically examine Kripke's arguments against Frege and their assumptions, and argue that a cognitive conception of meaning, dissociated from the descriptive theory of names, can accommodate the modal conception of meaning. I also argue that many of the basic assumptions behind Kripke's attack are doubtful from a Fregean point of view.

Chapter 1: Essentials of Frege's Logic[1]

Logic as a systematic theory of inference and of the validity of arguments begins with Aristotle, particularly in his theory of immediate inferences in his *De interpretatione* and mainly in his theory of the syllogism in the *Analytica Priora* (Aristotle, 1941). An inference consists of premises and conclusion, and it is valid when the conclusion logically follows from the premises, i.e. when the conclusion must be true if the premises are. An immediate inference is one in which one conclusion is inferred from one premise containing two terms, like "If all ravens are black, then everything that is not black is not a raven"; or "If all ravens are black then there are no ravens that are not black". We put here the inference as a conditional whose antecedent is the premise and the consequent is the conclusion, as Aristotle often does. Some of the most important of the immediate inferences were summed up in the famous "square of oppositions" of the Roman philosopher of the 6th century Boethius.

An Aristotelian syllogism is an inference in which a conclusion is inferred from two premises containing over all three terms, like "If all human being are mortal. and all Greeks are human beings, than all Greeks are mortal". Aristotle used "syllogism" for valid inferences; when an inference is invalid he often says "there is no syllogism". Aristotle realized that for a theory of valid inference some regimentation of language is required – there must be some standard forms of sentences. For various reasons into which we shall not enter here, Aristotle confined his logic to propositions of specific forms called "categorical". There are four such forms, each built by a subject and a predicate: an affirmative universal, like "All ravens are black"; a negative universal, like "No raven is black"; an affirmative particular like "Some ravens are black"; and a negative particular like "Some ravens are not black". In a syllogism one of the three terms is common to the two premises and is called the "middle term". Of the other two, one is the predicate of the conclusion and is called the "major" term, the other is the subject of the conclusion and is called the "minor" term. The premises in which they occur are likewise called major and minor. Syllogisms are characterized by their "figure"; there are four figures, each determined by the position of the middle term in the premises, i.e. whether it is the subject or predicate of the major or minor premise.

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[1] This chapter is a brief and general introductory survey. It is designed mainly for those who know elementary logic, but with some strain can be read also by others.

Aristotle enumerated the valid syllogisms in each figure. Actually, without calling it that, Aristotle devised what is perhaps the first axiomatic system, in which some valid syllogisms are set up as axioms and the rest are derived by some rules. They were later classified also according to their "mood" in each figure, where a mood is determined by the categorical character of the premises and the conclusion. Rules were set up for identifying the valid ones.

As regards the meaning of the categorical propositions, a problem that preoccupied logicians up to Frege is the problem of "distribution": what does a categorical proposition apply to? To what does it ascribe something? In "All As are Bs" we ascribe B to every A. "A" is then said to be distributed in the proposition. But nothing is ascribed to every B, so B is un-distributed. Particular propositions like "Some As are Bs" pose sever problems of distribution , as do also negative ones.[2] Rules for the validity of syllogisms were also set up in terms of distribution.

Aristotle not only discovered the general problem of the validity of arguments and inferences and some basic principles of coping with it, but developed logic to the point that in many areas with which he dealt, for more than two thousand years philosophers and logicians dealt in logic mainly in studying and interpreting his theory – sometimes with some technical and didactic improvements. We shall not deal here with the rich history of the subject,[3] but mention that at late 18th and early 19th centuries some logicians and mathematicians made significant contributions, with many innovations and significant changes of the Aristotelian theories. Notable among them is George Boole, who developed, with impressive mathematical acumen, algebraic calculi in which one could present and prove many inferences, including the Aristotelian syllogisms, as well as inferences that from a modern point of view are regarded as belonging to the propositional calculus. Denoting "the universe of discurse" by "1" and the empty class (or concept) by "0", Boole denoted the complement of a class A by "1-A", the union of A and B (assuming them to be mutually exclusive) by "A+B" and their intersection by "AB". Propositions were presented by equations; the Aristotelian categorical "All A are B" for instance was written as A(1-B)=0, and "Some A are B" as: AB≠0 (for which Boole invented the somewhat unclear "AB=ʊ").

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2 For a critical discussion of some of these issues see e.g. Geach (1963).

3 Two classics are Bochensky and Kneale & Kneale.

Boole proposed a general view centered on the idea of the algebraization of logic, in which algebraic equations and calculi could get various interpretations, in one of which they could express propositions and inferences of one area, in another inferences of a different area. Boolean logic, as presented and developed by e.g. E. Schroeder was the dominant one in Germany prior to Frege.

All this changed with Frege's *Begriffsschrift* (Conceptual Notation) of 1879 (BS). Historically, since this book, as well as other writings of Frege's, was so poorly accepted and hardly read until the 20th century, the Boolean logic continued to reign even after its publication. But substantially the change is quite drastic. From the vantage point of Frege's theory, Aristotle's syllogistic theory and its later developments by medieval sages, and later by Hamilton, De-Morgan and other logicians of the 18th and 19th centuries, were very limited in their expressive means and the extension of the implication relations and inferences they could set and explain. For Frege, not only were the restrictions to categorical propositions seen as arbitrary, and were lifted, but his account of the structure of propositions and their meaning (content) was completely novel and made the problems pertaining to the traditional notions of subject and predicate, the distribution of terms etc. artificial and irrelevant to the main task of logic.

Boole's algebraic calculi and the algebraization of logic he proposed by their means were a logical and mathematical achievement and a significant deviation from the Aristotelian conception. But from a Fregean point of view they were deficient in their philosophical principles and quite limited in the scope of their application and in their explanatory power, particularly with regard to the logic of relations and of propositions containing "iterated quantification" – roughly, when a quantifier like "all" or "some" applies to a content that includes other quantifiers. Without getting here into details, that are quite technical, the following example, which bothered already the scholastic sages can indicate the nature of the first problem. The inference saying that if anyone who owns a donkey beats it, then if Bile'am owns a donkey, he beats it, is certainly a valid one. It is admittedly not an Aristotelian syllogism, for "Bile'am owns a donkey" is not a categorical sentence. But this is a relatively minor problem, for one can extend the framework of categorical sentences to include also this one, as was indeed done shortly after Aristotle.

However, a more serious problem is that even within such an extended syllogistic system there was no systematic way to prove and explain the validity of the inference.

One of the main obstacles here was that a subject-predicate analysis, as was the norm in Aristotelian logic, in which the generality (expressed in the antecedent) was conceived as part of the subject or the predicate, cannot adequately express the crucial fact that it is the same donkey that is spoken of in the antecedent, saying that someone owns it, and in the consequent, saying that he beats it. This is also the case with the more advanced calculi of Boole. Other inferences involving what is now called "nested quantification" also posed insurmountable problems to the Aristotelian logic. An example is an inference like "If there is a cause to everything, then everything has a cause". These and similar problems with numerous inferences do not lend themselves to natural solutions within the confines of the Aristotelian conception. But they are easily solved in the Fregean logic and the quantification analysis it propounded. Frege also pointed out many formal deficiencies in Boole's definitions and in his proof procedures. He also claimed that the use of algebraic terms for logical notions misleads more than it helps. since their meanings in the Boolean calculus is different from their algebraic one.

Moreover, Frege thought that philosophically one should not see the Boolean calculi as logic, and that the very idea of the algebraization of logic is bogus and reverses the true order of things, for algebra should be based on logic rather than the other way around. The Boolean calculi seemed to Frege not more than artificial means of solving certain problems, but not a systematic theory of determining and accounting for inferences and implication relations of propositions. In one place he even remarks sarcastically that these calculi solve artificial problems that were invented just for the sake of being solved by the calculi... In any case they should not be regarded as logic as Frege conceived of it, namely as a systematic theory of the general laws of thought based on exposing the logical structure of propositions.

Logical structure for Frege is two-fold: on the one hand it determines the constituents of a proposition and how its meaning is built up by their means; on another, it determines and explains the implication relations of the proposition. This two-fold character of his conception of logical structure has deep and rich consequences, and is a characteristic mark of Frege's view and of its revolutionary significance. It is easy to miss this, for it was sort of absorbed and became common to most logical systems after him. It also manifests one of the important aspects of the difference between the Fregean and the Aristotelian (and Boolean) conceptions that were quite dominant up to him: An Aristotelian inference is based on relations between concepts (the terms of the

categorical sentences). For Frege, in contrast, the starting point is not concepts and their relations, but propositions. The validity of an inference is based on the implication relations between propositions as determined by their structure.[4]

The Formal Achievement of *Begriffsschrift* (BS)

It is quite common to say that modern logic began with Frege's BS. Frege set there up a formal language in whose framework he developed, in an axiomatic way, what is now called predicate calculus (of second order), including an axiomatic presentation of the logical theory called now predicate calculus of first order with identity, which is in fact complete (in the technical sense that any logical truth that can be formulated in that theory is provable there). In this axiomatic system Frege identified basic laws or logical truths (axioms) and formulated rules of inference or rules of proof, in a way that enabled in principle to prove any logical truth in that language. The axioms are not provable, according to Frege. Their status as logical truths, and their justifiability (in spite of not being provable) pose genuine problems. We shall deal with it in a later chapter.[5]

This system is in fact the system of elementary logic still taught and practiced today. The main difference between the logic of BS and modern approaches (particularly since the 1930s) is the absence in the BS logic of a formal semantics. The notions of interpretation, model, satisfaction, truth in a model etc. do not appear in BS. From this perspective Frege could not prove the above completeness, and in fact could not even formulate it. His logical system is not a formal calculus that can get various interpretations in various models, but a meaningful language in which the logical structures of the contents expressed are conspicuously presented (we shall expand on this in the next chapter). This language and the logic it contains is by far richer than the Aristotelian logic and the Boolean calculi, studied and practiced up to Frege, and it includes as special restricted cases whatever could be expressed by them, and much more. In particular it enabled to represent systematically and easily the logic of relations and of propositions containing iterated quantification, like "Although there is no biggest number, every number has a bigger one", or "Anyone who owns a donkey is envy of anyone who owns a stronger one".

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4 I expanded on these points in Bar-Elli (1985), here chapter 2.

5 I have expanded on this in "Analyticity and Justification in Frege". *Erkenntnis*: Volume 73, Issue 2 (2010), page 165-184. Here chapter 7.

This great achievement was almost a byproduct of Frege's main interest in the foundations of arithmetic: What is the basis of the validity of the propositions of arithmetic? Are they based on experience, or on special intuition, or perhaps on logic itself, which is the basis of all thinking? In the technical terms introduced by Kant the question was whether arithmetic is synthetic (as Kant thought) or analytic. These are important questions that have far-reaching consequences beyond the special concerns with arithmetic and they had preoccupied philosophers since the beginning of philosophy.

For a serious consideration of these problems, and from a profound conception of the essence of logic as a theory of justification that lies at the basis of our notion of objectivity, Frege developed a logical language that by far surpassed, both in extension and in precision, whatever was done in this area before. In doing that he created in fact modern logic. On this basis he thought that he could show that arithmetic was analytic, i.e. that with the help of appropriate definitions, all couched in purely logical terms, it is derivable from logical laws alone. It is in fact a developed branch of logic. By this, Frege propounded his "logicist" thesis, which, after BS, he continued to advocate and develop in exactness and responsibility that was not matched by anything before (and often also after). Since its birth this thesis faced serious difficulties mainly due to the paradoxes of naive set theory, and due to the ensuing difficulties concerning the possibility of regarding a set as a logical concept. Indeed towards the end of his life Frege himself gave up the thesis. During the 1930s Goedel's theorems enhanced and deepened these difficulties, but some of its aspects continue to challenge logicians and philosophers even today.

The basic ideas of the logic of BS remained valid in Frege's later thought, but on certain specific issues it contains various changes and developments of these ideas. These changes were introduced in a systematic way into the language of his magnum opus *Gundgesetze der Arithmetik* (*Basic Laws of Arithmetic*, henceforth BL). Beyond the enormous technical achievement of this work, they include, inter alia, 1) The systematic distinction between reference (*Bedeutung*) and sense (*Sinn*), which replaced the former undistinguished notion of content (*Inhalt*) of BS. 2) The conception of concepts and functions as the references of predicative expressions. 3) The conception of concepts as functions to truth-values (The True and The False) and of these truth-values as logical objects. 4) The introduction of the extension operator for forming the

extension of a function, and the conception of these extensions as logical objects (in fact, classes). We shall therefore first present here the basics of the logic of BS and then add some remarks on the later changes.

Frege was explicit that the language of BS is a logical language designed for the expression of genuine contents, not as a formal calculus of the sort of Boole and Schroeder. The Language of BS includes Greek letters functioning as constants, German (Gothic) letters as variables (also over functions), Latin letters as forming universal closures. It contains four logical constants for the conditional, negation, identity and universal quantifier. We shall present here some of the basic ideas using for this purpose only Latin letters and a notation common today.

At the basis of the language stand the atomic proposition consisting of an n-adic predicate (a capital letter) followed by n occurrences of singular terms (small letters), e.g. Fa, Rbc, Bxab. These atomic sentences are symbols for contents that would be expressed in English by e.g. "Moses is tall", "John is bigger than jack", "x is between Leeds and London". These atomic propositions[6] can combine into compound ones by means of propositional functions like the conditional (If... then---), negation, etc. In this way we get sentences like ~Fa (It is not the case that a is F), Fa→ Rbc (If a is F then b stand to c in the relation R). Another logical function, which is the main innovation of BS is the universal quantification (x)Fx (for any x, it is F). With this quantification a genuine use of the variables (x, y, z) is introduced. These logical functions can be applied in principle to any sentence – including those compounded by these very functions. In this way we can get compound sentences like (x)(Fx → ~Gx) (whatever is F is not G).

By means of the universal quantifier Frege showed how to express existential propositions like "There are Fs" or "There is at least one F": ~(x)~Fx (It is not the case that everything is not F). In many modern systems we use for this the "existential quantifier" and write $\exists xFx$. (Frege didn't use this quantifier and, as said above, showed how to define it by means of the universal one, so that $\exists xFx \equiv$ ~(x)~Fx.) Of special importance is the application of the quantifier on binary predicates (or relations) in sentences such as (x)(Hx → ~(y)~Fyx) which is the form of sentences like "Everybody has a father", "Any number has a successor" etc. "Anyone who owns a donkey beats it" can now be expressed thus: (x)(y)(Dy&Oxy → Bxy) (where Dx is x is a donkey, Oxy is

[6] I shall not be strict here about distinguishing sentences from propositions

x owns y, and Bxy is x beats y, it can be read: for any x and y, if y is a donkey and x owns y, x beats y).

The deductive system constructed in this language is based on nine axioms and two inference rules – *modus ponens* (see BS §6) and the rule of substitution. The first allows to derive the consequent from a conditional and its antecedent. The second – to substitute any sentence for a variable (but see below). In general, this system of BS is complete, except perhaps for the fact that Frege does not explicitly present there the rule of substitution, which he uses throughout quite freely. It should be also noted that Frege remarks that different logical constants and different axioms could be chosen so that the ones in his system could then be defined and derived.

The axioms in his system are the following (to remind, we use a modern notation in which the arrow is the conditional (if...then---), the tilde is negation, a variable between parenthesis is the universal quantifier. Parentheses over complete sentences indicate scopes in the familiar way. A variable within the scope of a quantifier containing it is called bound; otherwise – free. A sentence with a free variable is an open sentence; otherwise – a closed one. Thus the fourth axiom in the following list can be read as "if q then p, then if not-p then not-q"; the last axiom says that if a concept (or function) applies to any object, then it applies to c. In the seventh axiom the whole formula is the scope of (x); in the ninth – the whole formula is the scope of the conditional. To the right of each axiom we indicate its number and place in BS (where it is written if Frege's notation to be explained below)

$p \rightarrow (q \rightarrow p)$	(1 at the head o f §14)
$(r \rightarrow (q \rightarrow p)) \rightarrow ((r \rightarrow q) \rightarrow (r \rightarrow p))$	(2 at §14)
$(r \rightarrow (q \rightarrow p)) \rightarrow (q \rightarrow (r \rightarrow p))$	(8 at the head of §16)
$(q \rightarrow p) \rightarrow (\sim p \rightarrow \sim q)$	(8 at the head of §17)
$(\sim\sim p) \rightarrow p$	(head of §18)
$p \rightarrow (\sim\sim p)$	(head of §19)
$(x)(y)(x = y \rightarrow (F(x) \rightarrow F(y))$	(head of §20)
$(x)(x = x)$	(head of §21)
$(x)F(x) \rightarrow F(c)$	(head of §22)

The rule *modus ponens* allows to derive the consequent β from a sentence of the form $\alpha \rightarrow \beta$ and its antecedent α. Thus, it allows to derive $(q \rightarrow p)$ from the first axiom and p. In BS Frege does not formulate his sweeping substitution rule. But from his actual

substitutions in the proofs and from his remarks on allowed substitutions (which he probably regarded more as a convention of his notation than as a genuine rule of inference) one can conclude that besides simple substitutions of a sentence for a sentential letter (as in his explanation to 3 in §15), and of a singular terms for a (free) singular term, he actually adopts a rather sweeping rule that can be phrased thus: In any sentence containing Fx, where F is free, one can substitute any predicate or open sentence on x (where x is free) for all occurrences of Fx. In fact, Frege implicitly uses another rule allowing to derive $p \rightarrow F(c)$ from $p \rightarrow (x)F(x)$ (see §11 of BS, and Kneale and Kneale 489).

This is a complete basis of first-order predicate logic with identity, and Frege proves many basic theorems of it. He explains how to define various truth-functional connectives by means of the conditional and negation, and how to express existential propositions by means of negation and the universal quantifier, he defines the scope of a logical operator and explains the benefits of using quantifiers with different scopes, including those where one contains the other (nested quantification). Moreover the sweeping substitution rule Frege in fact adopts expands his system into a second-order logic (not complete of course, but consistent as far as is known).

Within this system Frege defines in the third part of BS some important notions, and proves central theorems of what he calls "theory of sequences". On their basis he formulates and proves a version of the principle of (mathematical) induction as a logical truth. Frege ascribed supreme importance to this, for it showed, he thought, that it is possible to define the basic concepts and to prove the basic properties of the natural numbers and of arithmetic in "pure" logic, with no appeal to "intuition". The special significance of this pertains to the great philosophical dilemmas about arithmetic to which we have alluded above: On the one hand, arithmetical truths are knowable a priori, independently of experience, while on the other, they do not seem to be purely logical. What then can be the basis of their validity? In the history of philosophy there are various efforts to answer this problem, and we shall not enter into them. Frege's answer is, very briefly, to deny the second horn: when logic is properly conceived, arithmetic is ultimately logic. Therefore, its a priori nature is not more enigmatic than that of logic itself.

This topic, which was undoubtedly a basic motivation of developing the logic of BS, is not discussed in detail in that book. Frege dealt with it, on the basis of his logic of

BS, in much greater detail, though in a general and not strictly formal way, in his next great book *Grundlagen der Arithmetik* (*Foundations of Arithmetic* (FA) of 1884, and later, in a more formal manner in BL of 1893. In both the language, definitions and deductive system of BS are the basis. In fact, as was recently shown (mainly by G. Boolos), the logical system of BS with one additional premise suffices to prove the basic properties of the natural numbers (the so called Peano axioms) and to reduce arithmetic to logic. The additional premise – the so called Hume's principle – says roughly, that two sets are equinumerous (have the same number of members) if and only if there is a one-one correspondence (mapping) of the one onto the other, i.e. if there is a function mapping each member of the one to one and only one member of the other (and vice versa). In other words the principle says that the number of Fs (N(F)) equals the number of Gs (N(G)) if and only if there is such a function (≈), We can therefore symbolize the principle by N(F)=N(G) ≡ F≈G. Due to a passage in which Frege mentions Hume in connection with this principle it is currently often called Hume's principle. Frege formulated the principle in FA and sketched informally the essentials of the above reduction. A more formal presentation is provided in BL, but he thought that the principle, though quite intuitive and reasonable, is not of the status of a logical axiom. He therefore proved it on the basis of an axiom that seemed more basic, and which he regarded, though with reservations, as a logical axiom. This is the notorious basic law (axiom) V of BL The axioms says roughly that the extension of a concept is the same as the extension of another concept if and only if all objects to which the one applies are objects to which the other applies as well. However, by implying that any concept has an extension, it turned out that this axiom is too strong, and made the system of BL inconsistent. This inserted what Frege later regarded to be a fatal blow on his logicist project.

Frege proposed in BS some important definitions that were useful for his proofs, and significant for his logicist project. The notions of sequence and order may seem to require special intuition (as e.g. Kant thought) and to lie outside the province of logic. However, the following definitions may exemplify how Frege thought to define them by purely logical terms. We write here some of the main ones (in a formulation that expresses the basic idea, though not strictly Frege's). Following Frege, the letter "f" is used here both for a sequence (*Verfahren*) and for a relation generating the sequence. Frege's idea here (in the third part of BS) was roughly that a sequence is generated by

applying a binary relation to an object and then continue applying it to the results of previous applications.

- A property F is **hereditary** (*sich vererbt*) in a sequence f if and only if F is a property of whatever stands in the relation f to what is F [(x)(Fx → (y)(f(x,y) → Fy)]. We shall symbolize it as H(F,f). For example, if the relation f is to be the son of, then the property of being tall is hereditary if and only if whoever is a son of a tall person is tall. (see BS §24).
- y is a **follower** of x (*folgt auf x*) in the sequence f (or: x is ancestor of y in f) if and only if y has any hereditary property F that is a property of whatever stands in the relation f to x. We can write the condition as: (F)(H(F,f)→(z)((f(x,z)→Fz)→Fy) (see BS §26).
- z **belongs** (*gehoert*) to the sequence beginning with x if and only if z is a follower of x in f or identical with it. (see BS §29)
- A relation f is **single-valued** (a function) if and only if any two things related to something in the relation f are identical [(z)(y)(f(y,z)→(x)((f(y,x)→x=z)] (BS §31).

By means of these (and other) definitions Frege proves some important theorems on sequences and functions. We shall write here some, using the following abbreviations: H(F,f) for F being hereditary in f; A(x,y) for x being ancestor of y in f; I(f) for f being a function; B(x,f-y) for x belonging to the sequence f beginning with y. Again, the following formulations are not entirely precise, but are essentially Frege's.

- Any consequent of x in f has all hereditary properties of x in f. [F(x)→(H(F,f)→(A(x,y)→Fy))] (81, §27). If f is the sequence of natural numbers, this is in fact the Fregean formulation for the principle of (mathematical) induction.
- Whatever stands in the relation f to a consequent of x is a consequent of x (96 §28). [A(x,y)→(f(y,z)→A(x,z)]
- The property of belonging to the sequence beginning with x is hereditary in the sequence. [H(B(y, f-x),f)] (109, §30)
- Whatever stands in the relation f to something belonging to the sequence f beginning with z, precedes z in the sequence or belongs to the sequence beginning with z [B(y,f-z)→(f(y,v)→(A(v,z)v(B(v,f-z)))] (111, §30)
- Whatever belongs to the sequence f beginning with x follows x in the sequence or begins a sequence to which x belongs [B(x,f-z)→(A(z,x)vB(z,f-x))] (114 §30).

- If a relation f is single-valued (is a function), then any follower of y in the sequence f belongs to a sequence beginning with whatever stands in the relation f to y [I(f)→(f(y,x)→(A(y,m)→B(m,f-x)))] (124 §31).
- If z and y are followers of x in a sequence generated by single-valued relation f (function), then z is a consequent of y in the sequence, or y belongs to the sequence f beginning with z. [I(f)→(A(x,m)→(A(x,y)→(A(y,m)vB(y,f-m))))].

Frege explicitly remarks in BS that the proofs of these propositions are especially important not because they establish facts that we didn't know or that we doubted before, but because they do it on the basis of purely logical principles and definitions, without any reliance on intuition. Such a claim may not be particularly surprising with regard to classical notions and truths (like identity, propositional connective and first order quantification). But Frege assigned them particular significance concerning the notions of order and following in a sequence, which, on the face of it, do not look like logical notions, but as requiring intuition. This is therefore of supreme importance, for, on Frege's view, these notions are at the basis of arithmetic. At §81 of FA Frege says, relating to the above definition of following in a sequence: "Only by this definition of following in a series is it possible to reduce the argument from n to (n+1), which on the face of it is peculiar to mathematics, to the general laws of logic".

In chapter IV of FA, on the basis of this logic, Frege outlines his way of reducing arithmetic to logic. Briefly sketched it is thus: He defines the notions of number and of following in the sequence of numbers, and shows how to define the specific natural numbers 0. 1. 2, ... He proves the existence and uniqueness of the numbers as thus defined, and the basic axioms of arithmetic, including in particular that of (mathematical) induction, insuring that there are infinitely many numbers.

He begins by defining "the number of Fs" (the number belonging to the concept F, which we may symbolize by N(F)) as the extension of the (second-order) predicate "being equinumerous to F", where two concepts F and G are equinumerous if there is a 1-1 relation between their extensions (§68-71), i.e. if there is a relation R that to any object in each one of them relates a unique object in the other. That is, if (x)((Fx→∃y(Gy&xRy&(z)(Gz→(xRz→y=z)) and (x)((Gx→∃y(Fy&xRy&(z)(Fz→(xRz→y=z)).

Having this notion of the number belonging to F, Frege defines n is a number if and only if there is a concept F such that n is the number of Fs (the number belonging to F). He proves that if two concepts are equinumerous, their respective numbers are the same (§73).

The natural numbers are then defined (from §74): 0 is defined as the number belonging to the concept of being different from itself (0=N(x≠x)), and its existence and uniqueness are proved. 1 is defined as the number belonging to the concept of being equal to 0 (1=N(x=0).

In §76 Frege defines that n is the immediate follower (successor) of m in the sequence of numbers by: "There is an F such that n belongs to F, and there is an x which is F such that m belongs to the concept of being an F different from x" (n=N(F)&m=N(F&≠x). For this, one needs provide a concept to which the successor of a number belongs, and Frege suggests "being a member of the sequence of numbers up to n", which is defined on the basis of the above logical definition of the ancestral relation (§79; see also §82 for a proof that this is the desired number). Frege elaborates on the objectivity of this notion of following in a sequence, and that it does not depend on intuition (§80). He also defines the infinite number of natural numbers (§84-6). This ends our sketch of his outline program in FA, to be highly elaborated in BL.

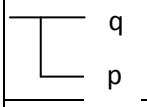
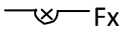
As said before, the possibility of basing arithmetic on pure logic was of special philosophical importance, for arithmetic was conceived (at least since Kant) as a paradigm of a system of truths that, on the one hand, are knowable a priori (with no relying on experience) and ,on the other, are synthetic – extends our knowledge and are not merely logical tautologies. This combination looks enigmatic: we may grant the idea that a logical truth is knowable a priori; it is much more difficult however, to come to terms with thinking so about a synthetic one. Frege's achievement, besides its formal importance, enabled him to deny that arithmetical truths are synthetic and to claim that they are based on logic alone (though this involved changing the conception of logic and its limits that was common up to him). In this way he was able to make a crucial contribution to exploring the problem of the source of validity of arithmetical truths.

The Logical Notation of BS

It seems that nothing caused misunderstanding, and in fact relative ignorance of BS and of Frege's logical writings in general more than his unique notation. Frege thought much and deeply about his notation, and its benefits and advantages over alternative systems

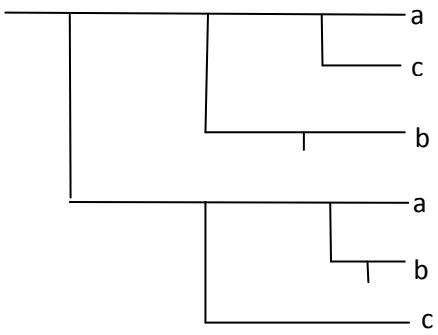
(like the "linear" Peano-Russell one) are quite obvious. The main advantage of Frege's system is that without the use of parentheses, the logical structure of a proposition (formula) – its main operator, and the scope of each operator within it – are identified at a blink. This is due to the special tree-like structure of the "content strokes" (see below). Likewise the specific "contents", the atomic sentences, written at the right hand side of the formulae, are easily recognized. In spite of these advantages, this system has not been accepted – in fact, no one used it except Frege. It still deters many of reading him.

The notational system he uses is clearly explained in BS. We shall present the basics here and "translate" it to the modern more common one.

Frege's Notation	Customary Modern Notations	Meaning
	p→q p ɔq	Conditional: If p, then q
	~p ¬p	Negation: Not p; It is not the case that p
	(x)Fx ∏xFx ∀xFx	Universal Quantification: For every x, Fx Everything is F

To the left of each sentence there is a horizontal line called "the content stroke". The conditional of two sentences is marked by a vertical connecting the left ends of their content strokes. Thus the conditional in the table consists of content strokes to the left of p and of q, a vertical (the conditional line) connecting their left ends, and a content stroke for the conditional. The negation of p consists of a content stroke to the left of p, a short vertical at its left end (the negation line) and a content stroke for the negation. Similarly the universal quantification consists of a content stroke to the left of "Fx", a concavity on x, and a content stroke for the whole quantification. Frege also writes a vertical at the utmost left of the main content stroke – the "judgment stroke" – (not shown in the table), indicating that the content is judged or asserted. Two strokes at the left indicate a definition.

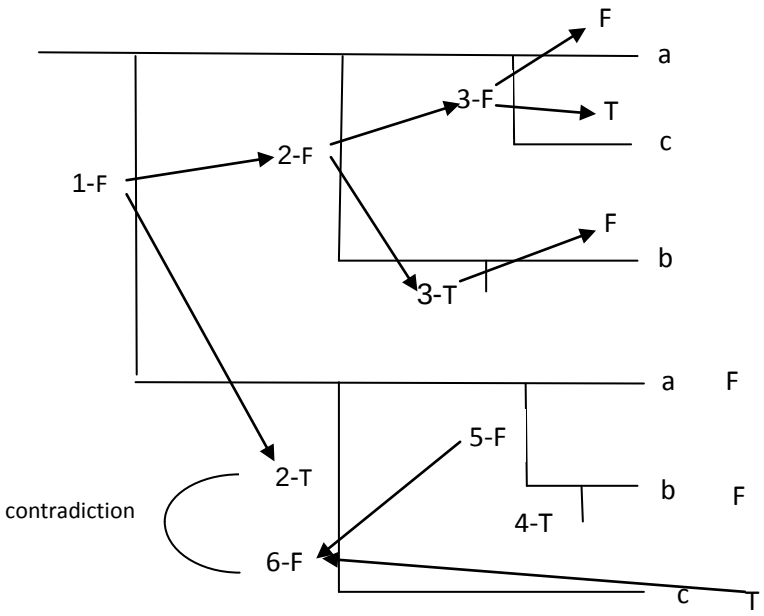
The great majority of the formulas in BS are conditionals, in which the antecedent, as explained above, is written below the consequent. A reader used to read conditionals with the antecedent at the beginning may wish to read the Fregean formulas from bottom up. The following is proposition (35) from §18 of BS



It can be read: If c then if not b then a, then if not a then if c then b. Less mechanically it reads: If c implies that if the negation of b implies a, then the negation of a implies that c implies b. In a customary modern notation it is: (c→ (~b → a)) → (~ a → (c → b)). Frege proves this formally, but it would be noted that in his notational system and the explanation with which he accompanies it there is an anticipation of the basic idea of "semantic tableaux" customary in modern logic, although in Frege it does not appear as a systematic semantic theory. A Fregean formula has a tree-like structure from left to right. One who is used to reading such structure from top down can see this by turning the Fregean formula 90^0 clockwise. For non-quantified formulas this can be explained (again, roughly and inaccurately) as follows: The whole formula is the tree-trunk; each conditional is split to its components (the antecedent and consequent), and a branch in the tree is a sequence of components in which the "lower" ones are in the scope of those above them. In the explanations with which he accompanies his formulas, both in BS and in BL, Frege specifies what each formula (and sub-formula) excludes (or denies), namely, what is implied by its falsity.

Thus, the reader can mark with F (false) the main operator and continue marking the implications of this along the branches. If this process brings to a contradiction – a place where T and F appear together – this means that the assumption that the entire proposition is false is impossible – cannot be true. i.e. the proposition must be true. In this way the reader can not only grasp the meaning of the formula, but also prove its being a logical truth, in showing that assuming it false is impossible, leads to a contradiction.

(35)



Let us exemplify it again with respect to the above proposition 35 : The letters T and F indicate truth and falsity. We begin at stage 1 by assuming that the entire formula is false and mark it with F to the left of the main operator – the most left vertical. We then mark with arrows what this implies at stage 2 namely T at the antecedent (bottom) and F at the consequent (top). We then continue at stage 3 with the implications of the F of the consequent of stage 2, namely T at the negation of "a" and F at its consequent, which implies that "c" is T and "a" is F. We then assign these values to these letters at the bottom, indicating that they were got from other places by writing the values at the right of the letters. Carrying on the implications of these we get at stage 4 T to the negation of b, and at stage 5 F to the conditional, which implies F to the antecedent of the main conditional. But this was already assigned T at stage 2. This is a contradiction, which means that our initial assumption that the formula can be false was impossible. This proves that it is a logical truth.

Some Later Developments

As said above the language of BS was the basis of Frege's logic throughout his entire career. But in the early nineties he introduced some changes and additions – some, of great philosophical significance. The main one is the systematic distinction between sense and reference and its insertion into his logic. We shall expand on it in subsequent chapters. Another is the introduction of his extended notion of function as the reference of predicative expressions, and its distinction from its extension. Frege largely extended the mathematical notion of a function to include in fact everything in the realm of

reference which is not an object. Thus, functions for him are not just mathematical functions like x plus y or sin x that map numbers to numbers (or more generally objects to objects) but also concepts and relations. These he regarded as mapping (sequences of) objects to truth-values. Thus being odd, for example, maps 3 to the truth and 2 to the false, and being smaller maps (2, 3) to the truth and (5,3) to the false. Likewise, being in France maps Paris to the truth and London to the false, etc.

For the extension of a function, the range of its values or the objects it maps to the truth, Frege introduced a special operator that when attached to a functional or predicative expression forms a name of its extension. Functions should be distinguished from their extensions. Functions (including concepts and relations) are references of predicative expressions, and are essentially incomplete, in analogy to the incompleteness of their respective expressions. Extensions of functions are in fact classes, which Frege insists are objects, and therefore cannot be the references of predicative expressions. We shall symbolize the extension of Fx by x^Fx. This operator is governed by a special logical principle introduced already in the article *Funktion und Begriff* (*Function and Concept*, henceforth FC) of 1891 and presented as an explicit axiom in the system of BL of 1893. It states that the extensions of two functions are the same if and only if every object that falls under the one falls also under the other. Using the above notation for extension it says $x\hat{}Fx=x\hat{}Gx \leftrightarrow (x)(Fx \leftrightarrow Gx)$. This implies that every concept has an extension, which, as Russell pointed out, proved to be inconsistent and, as Frege was quick to realize, drew a fatal blow on his logicist project.

In BL Frege also introduced a special function, symbolized by a diagonal to the left of an extension-name (\ξ) that is defined as follows: If there is a unique, one and only one object which is identical with the argument (the reference of the name) the value of the function is this object; if there isn't, i.e. if the extension is empty, or consists of more than one object, the value of the function is the argument, namely the extension itelf (see BL §11). Obviously, in any case there is a value, and all "empty names" refer to the same thing – the null extension. Frege points out that this function is similar to many uses of the definite description of natural language. He also points out in this connection that these uses display the existence and uniqueness conditions, namely that a genuine use of such a description presupposes that there is an object, and only one object that satisfies the concept referred to by the predicate concerned. This, and its relation to Russell's famous theory of descriptions will be the topic of a subsequent

chapter (5). On the basis of this descriptive operator Frege also defines what he calls "concatenation". The general definition is rather complicated but the basic idea is simply that the concatenation of "a" and "f" (symbolized by "a∩f") is a name referring to the value of the function f for the argument a (see BL §34).

Logic and Philosophy

We cannot end such a survey, though only rough and general, without mentioning that besides the revolutionary novelty of its technical-formal aspects, Frege's system was constructed from an exceptionally wide and deep philosophical view, both with regard the nature of logic and its general status, and with regard particular topics in the philosophy of logic. We cannot enter here into a detailed discussion of these issues, some of which will be pursued in subsequent chapters. We shall just mention some such topics that are prominent already in BS: 1) Conceiving logic as the science of justification and objectivity and as its paradigm. 2) Conceiving the logical structure of a sentence as constituting its content, on the one hand, and its implication relations on the other. 3) The primary status of a complete sentence (or its meaning) and conceiving the meaning of its components as their contribution to that of the sentence. 4) Replacing the traditional conception that the basic structure of a sentence is that of subject-predicate by the conception of a function and its arguments. 5) Recognizing the various possibilities of a functional analysis of the same sentence (and content). 6) The distinction between content and judgment. and characterizing features of meaning that are not part of the content in its narrow sense. 7) The way categorical ontological distinctions like that of concept (function) and object are expressed in the structure of the logical language. 8) The nature of formal proof, and some aspects of the relationships between implication and formal proof.

On all these, some of which will be discussed in further chapters, and on many other points, Frege's logic, including its developments after BS, expressed deep insights. Let me mention here one famous and particularly important example.

Existence-sentences like "Some philosophers are rich" and their alleged equivalence to "There are rich philosophers" or "rich philosophers exist" posed a problem for which Frege's logic suggested a natural and radical solution, very different from preceding conceptions. In the Aristotelian logic existence, expressed by "some" in the above, is part of the subject of a categorical proposition. This attitude raises many problems. We shall mention two: a) Grammatically, "there is" is an inflexion of the verb

"to be" which functions usually as a predicate. One faces therefore a difficulty in explaining the equivalence between the first proposition and the second (or third). Is existence implied by the mood or the categorical character of a proposition, or is it a predicate expressing a property? a property of what? b) "There are philosophers" or "Philosophers exist" are propositions, while "Some philosophers" is not. So "some" is of a different logical category than "exist". How then can we account for the above equivalence?

A widespread conception in philosophy, which also has an intuitive basis is that existence is a property (mainly of objects). Many versions of the "ontological proof" of the existence of God, for instance, rely on it. Kant famously argued against this view and against the proof based on it, and proclaimed that "existence is not a predicate" (a property). If I have a bill of $100 in my pocket, it may have various properties – folded, old, in my pocket, of a certain color, weight etc. In each of these it may differ from other bills. But it is a mistake, Kant thought, that one of the properties in which bills may differ is existence. A bill that does not exist is not a bill that lacks a particular property that other bills have. Kant's alternative view presents severe difficulties pertaining to central issues in his philosophy like the kind of relationship between concepts and intuition (*Anschauung*) into which we shall not get here. And there are of course other approaches.

Frege's quantification theory changed all this. In BS Frege just shows how to express existential statements in his notation, using negation and the universal quantifier, so that "there are Fs" or "Fs exist" is expressed by ~(x)~Fx. Hence, his logical language does not provide for ascribing existence to objects, or to attaching "exists" to a proper name. And this is a view Frege persistently defended also on intuitive grounds (e.g. FA §§48-53). Since FA of 1884 and in his mature view existence is indeed a property, but a second-order one. It is a property of concepts or functions, namely to the effect that they are instanced (e.g. FA, §53), or as he would later put it, that their extension is not empty. He therefore agrees with Kant that existence is not a property of objects, but this does not exclude its being a property of concepts. This theory enabled Frege to propose easy and sharp solutions to many traditional problems pertaining to existence. It also enabled to conspicuously present logical connections between existence statements and others (like singular and universal ones). From this perspective Frege's theory of quantification was not only a revolutionary novelty in

logic, in the narrow sense, but provided a ground for a new conception of a basic ontological notion – existence.[7]

There are many problems connected with the meaning of existence or of analysing statements of existence, and we cannot even touch on that big issue. Besides the above mentioned problem of whether ascribing existence is a form of predication, another notorious one was that of non-existence: it seems that denying existence of something is impossible for that of which we deny existence must be something. Philosophers then proposed distinguishing existence from being, or various kinds of existence (even Russell of PoM was still convinced of the need to such distinctions). Many theorists still wish to distinguish e.g. between temporal existence, which seems to be a regular property, and a-temporal existence. Geach (1963, 90-91), for instance, distinguished between statements of existence (and non-existence) like some uses of "philosophers exist", "Centaurs do not exist", in which existence is a-temporal, and "Dinosaurs do not exist", in which existence is temporal and should perhaps be better phrased "no longer exist" (for a critique of Geach's two senses construal see Dummett, 1981a 386-390). Frege's quantification theory offered persuasive solutions to many of these problems, made some of these proposals impertinent, and allowed great flexibility in introducing all sorts of concepts by the basic distinction it laid between the quantificational base of the theory (the range of objects and functions it quantifies over) and its predicates.

[7] It is worth noting that in his early "Dialogue with Puenjer on Existence" (PW 53/60-67/75) it is Puenjer who construes existential statements like 'There are men' as being about concepts (no. 12 p. 53/60-54/61) – a view close to Frege's mature one. Frege presents there a view quite different from his mature one, according to which existence is not a property at all but a feature of the logical form of the sentence (62/69-70). He also argues there that existence is not a property of objects, which of course remained also his mature view.

Chapter 2: Frege's Early Conception of Logic

According to some historians of philosophy, Frege's revolutionary work in logic was a by-product of his concerns in the philosophy of mathematics, i.e. of his desire to found mathematics on a firm basis. Consequently, it is argued that Frege was not quite aware of the revolutionary character of his work in logic, nor of the novelty of his conception of logic: what kind of science it is, what are its goals and subject matter, etc. I believe that this last claim is wrong and that it derives, perhaps, more from Frege's cryptic style than from actual examination of the texts. Moreover, since the first claim about Frege's concerns with the foundations of mathematics, is, so far as it goes, correct, its consequence of putting his early work in logic in the shadow of his logicistic program, results in an almost total neglect of other aspects of his early conception of logic, which, , though perhaps less dramatic are still important in themselves and pertinent to a proper understanding of his philosophy at large.[1]

Frege's scattered remarks about the nature of logic, the significance and merits of his own work in comparison with earlier conceptions constitute a network of ideas whose connections with each other, though never explicitly and systematically presented in his writings, lend themselves, I believe, to such a presentation, and hence, the justification of my use, in the singular, of "Frege's Conception of Logic".[2] I shall confine myself in the

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[1] Apart from "intrinsic" questions as whether to regard this or that notion or axiom as "logical", it should be noticed that unless one gets clearer about Frege‘s conception of logic one can hardly make sense of such central topics in Frege's philosophy as the nature of analytic truth, the notion of objectivity, the nature and significance of what he regarded as defects of natural languages, and the reasons for his view that there are logical objects (whose existence is guaranteed by logic alone). Each of these deserves, of course, a discussion on its own, which I cannot carry out here.

[2] As M. Dummett has remarked, Frege used the word "*Logik*" to refer to what we would now call a theory of meaning. This, I believe, is true particularly of Frege's mature and late writings. Granted that, we may still ask about Frege's conception of logic in the narrower sense, which is more common today, and which Frege, perhaps, used in his earlier writings. It is part of the thesis of the present article, however, that Frege's earlier conception of logic already "contains the seeds" of

following presentation to Frege's early writings, particularly the *Begriffsschrift* (Conceptual Notation) of 1879 and some related articles of the early eighties. Where I shall refer to later writings I shall just use formulations which seem to me particularly happy for ideas already expressed in the earlier writings. I shall also try to stick as close as possible to Frege's own terminology and to avoid more modern formulations even where, at points, they seem quite suggestive. The discussion will be quite general and non-technical.[3] I shall also avoid comparisons to later philosophers even where they seem tempting (I'm thinking especially of the early Wittgenstein and Carnap).

Reference and Inference

Logic has been developed along two different lines: as a theory of inference, and as a theory of reference (or, meaning). The first is the more common and traditional one, and it is the characterization of logic to which Frege repeatedly appeals:

> To make a judgement because we are cognisant of other truths as providing a justification for it is known as inferring. There are laws governing this kind of justification, and to set up these laws of valid inference is the goal of logic ("Logic", PW, p. 3).[4]
>
> The task of logic being what it is, it follows that we must turn our backs on anything that is not necessary for setting up the laws of inference (ibid, p. 5).

The psychological ring of some of the terms in these and other formulations must not mislead us, for Frege emphasized more than anyone else that the rules in question are

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this broader notion of a theory of meaning. For, Frege's conception of logic as a *lingua characterica*, to be presented here, consists of the thesis that a theory of valid inference should be integrated in a more comprehensive theory of meaning.

3 This is a serious restriction. A more detailed and technical analysis would have to deal with some topics that I leave untouched, such as Frege's notion of function and its significance, the problem of alternative analyses of the same content, etc. I could not deal with these without describing Frege's actual logical system, which would change the whole character of the article.

4 Consult the list of references and abbreviations at the end. I have avoided discussions of the secondary literature which, in general, is concerned with the ostensive topic of this article in a casual and unsystematic way.

concerned only with the truth-dependencies of the statements involved, and that the actual psychological or physical processes of making these judgements and inferences[5] are not in question at all. It is in this context – of separating the normative, logical study of correct inference from the psychological and other empirical studies of the processes involved in inferring – that Frege repeatedly appealed to the objective notion of truth as characterizing logic: "[…] the laws of logic are nothing but the unfolding of the content of the word true" (ibid.).

This, evidently, is a central issue in Frege's philosophy. The distinction between concepts and ideas, so much discussed in *Grundlagen der Arithmetik* of 1884, the objectivity of thoughts (*Gedanken*), so emphasized in his later writings are all connected to this separation of the logical and the psychological, which is based on the objectivity of truth. It is not that psychological processes or states are not objective; they are as objective as any other empirical phenomena: "Psychology is only concerned with truth in the way every other science is, in that its goal is to extend the domain of truths" ("Logic", PW, p. 3). Frege's point is, rather, that the claim for truth involved in, and in fact constitutive of, the act of assertion cannot be adequately analyzed, or described in psychological terms, as he put it in a late piece:

> […] what logic is really concerned with is not contained in the word true at all, but in the assertoric force with which a sentence is uttered ("My basic logical Insights", PW p. 252/272).

In any case, correct inference is characterized in terms of truth, for it is one in which the truth of the conclusion, if it is true, is conferred upon it by the truth of the premises. And when the truth of a statement depends on that of another, the study of this relationship is the subject matter of logic ("Logic", PW 1-8).

The notion of truth is taken by Frege here, as elsewhere, as an elementary, irreducible, and intuitively well-understood. Logic's concern with this notion of truth is construed, therefore, in an entirely different way from the one to which logicians has

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[5] Following Frege I use the words "inference", "laws of inference" etc. in a rather loose way. As van Heijenoort and others have rightly indicated, Frege did not have the modern notion of model and of truth in a model, and was not aware of problems of completeness. He therefore did not distinguish between proof-theoretic inference and semantic implication, and any precizations in these matters with respect to his writings would be anachronistic.

been accustomed since Tarski's work in semantics and model theory. Truth, for Frege, is undefinable and its role in logic is very different in his conception from the one characterized in terms of the relationship of a language to its models (Heijenoort, 1967).

Now, there is in Frege's later philosophy another way in which logic is connected to the notion of truth. Here, truth is construed as one of the two "truth values" and logical notions can be roughly (and very imprecisely) characterized as functions whose range is the set of these two truth values. This, of course, is connected to the basic principle of Frege's theory of reference, namely, that the reference of a complex (and in particular a sentence) is a function (and where the reference of a sentence is in question – a logical function) of the references of its constituents. This conception may be strictly developed into a Tarski like semantics, and it is this conception to which I alluded in the rubric "logic as a theory reference".

Frege had never, as far as I know, characterized logic in these terms explicitly, but in some sense (which I shall make clear later on) it underlined his conception of logic. It reveals the rationale of the kind of constraints he imposed on an adequate theory of inference, of his various criticisms of alternative conceptions of logic, and of the merits he ascribed to his own system. Granted this, I shall argue that the great novelty, in fact, the revolutionary character of Frege's conception of logic lies in this dual aspect of his view: that logic is a theory of inference and a theory of reference (or of meaning), or in fact, that logic is a theory of inference by being a theory of reference. This conception, I believe, lies behind Frege's insistence (on which we shall elaborate in the sequel) that logic is a *lingua characterica* and not merely a *calculus ratiocinator*. One of the things this conception implies is that logical principles (axioms and rules of inference) should be derivable from the very structures of the sentences of the language. This may explain Frege's major task – that of constructing a language that will satisfy this demand (this is a necessary, not sufficient condition). It may also explain various things Frege says about logic in an almost self-evident manner: that logical principles should be universal, that they should be explanatory, not *ad hoc*, or arbitrary. It also explains a point of great importance to Frege: that in contrast to traditional conceptions, logic should start from the analysis of the complete sentence or sentential content and not from the analysis of "concepts and their relations" ("Notes for L. Darmstaedter", PW 253-258). Some of these points are stated explicitly by Frege himself, some emerge by examining his critique of

rival conceptions. It wouldn't be improper, therefore, to begin by a brief resumè of this critique.

Frege on the deficiencies of earlier logical systems

Much of Frege's conception of logic can be gathered from scattered remarks he makes on the subject in several early articles he wrote on his BS, in which he compared his own system with earlier conceptions. It is often said that Frege's main advance over previous logical systems – and that of Boole in particular – is the wider scope of the applicability of his BS, to wit, his treatment of generality and of embedding. This is, of course, true and it marks the major technical breakthrough of Frege's logical notation. But this technical point does not reveal in itself the full extent of the philosophical conception of logic that underlies it. If we examine Frege's critical remarks of Boole's logic (which was the most advanced and esteemed at that time) and of other conceptions, we shall see that Frege's view of the difference – in fact, the unbridgeable gap – between these systems and his own work was much wider and more penetrating.

Frege was, of course, aware of the limited scope of Boole's logic. He says, for instance, that Boole cannot adequately express particular and existential statements (CN 92, cf. PW 14).

He also criticizes Boole for various notational defects, as, for instance, the use of arithmetical operations which have a totally different meaning in the logical system, (CN, 92, 94) or for the ambiguous use of some expressions like identity (PW 14-16). He also criticizes Boole for the lack of some important definitions, and for the inadequacy of others, etc.

But all these points are, according to Frege's own view, of secondary importance. There are others, however, to which Frege seemed to attach much more importance, and which reveal the extent of his awareness of the nature of his revolution in logic. The three main points I have in mind are the following:

(I) Boole's division of his logical system to "primary" and "secondary" propositions is *ad hoc* and lacks explanatory power;

(II) Boole's system, like all previous ones, incorporates a conception according to which concepts are logically prior to judgements, or to propositions;

(III) Boole's logic, like all previous ones, is at best *calculus ratiocinator*, whereas a genuine logical system should be *lingua characterica*.

I'll discuss the first two points in turn, leaving the third, which is the main one, for the next section.

(I) Boole, in his *The Laws of Thought*, followed a traditional division of logic into two compartments: the first treats of what he calls primary propositions, which are basically the Aristotelian categorical propositions. These propositions state, according to Boole, relations between concepts, or classes, and the Boolean operations have, accordingly, more or less their customary set-theoretic interpretation. Thus "A+B" denotes the concept of being either A or B, or the class of things which are either A or B assuming, as Boole did that A nd B are mutually exclusive). "AB" denotes the concept of being both A and B etc. Judgements are formed by equalities, or inequalities between these expressions and the two constants – 0 (denoting the null class) and 1 (denoting the universal class).

The second compartment is that of "secondary propositions", which, using the same operations, state relations between propositions "considered as true or false".

Boole noticed a "remarkable analogy" between the formal laws of these two classes, which he tries to explain by reducing secondary propositions to primary ones. The reduction is carried through a questionable interpretation of secondary propositions as primary ones which are about classes of moments of time: "If p then q" for instance is thus "reduced" to the categorical: "all moments of time in which 'p' is true are moments of time in which 'q' is true".

Frege rightly complained that this manner of reduction is *ad hoc* and artificial and does not reveal, or explains the real unity of the logical field. He contrasts it with his own treatment where both kinds of propositions are formed by the same logical operations, so that the distinction between them is effected by a distinction of the relative scopes of the operators, and "in this way, in place of the artificial Boolean elaboration, an organic relation between the primary and the secondary proposition is established" (CN, p. 99).

This point of Frege's critique of Boole reveals, I believe, three components of his conception of logic, to which he alludes also on other occasions. These are the universality, or generality, of logic, its homogeneity, and its explanatory power. I shall discuss these three points in turn.

Frege often appeals to the universal character of logical principles and logical notions. Logical principles are marked by being universally valid; they are universal truths. Their validity is not restricted to a particular domain or subject matter. This

universal character of logical truths had great epistemological significance for Frege. They are truths on which all knowledge is based. (This theme appears already in the preface to BS, and becomes, of course, prominent in *Grundlagen*.)

"The task we assign logic is only that of saying what holds with the utmost generality for all thinking, whatever its subject matter" (PW 128/139). The universality of logic means, first and foremost, that as far as logical matters are concerned there can be no area or subject that has its "own logic", that logical principles and logical notions cannot be separated according to their applicability, or validity to subject matters. This, as he makes clear in *Grundlagen*, was one of his motivations for reducing arithmetic to logic. He wanted to prove, what for him was evident a priori, that arithmetic truths do not rely on some special logical principle (the principle of induction) which is not universally valid. One should also recall here that the universal applicability of arithmetics was for Frege a sure sign of its "in-principle" logical character (*Grundlagen*, Preface iii-iv).

Frege nowhere explains, so far as I know, whence does logic get this universal character, how does it come to consist of truths which are independent of any particular subject matter and are universally applicable. But some of his remarks suggest that logical truths derive their validity from the very nature, or structure of thought itself. In explaining the universal character of logic (and arithmetic) in the preface to *Grundlagen* he remarks: "Thought is essentially the same everywhere, there are no different kinds of laws of thought for different objects" (*Grundlagen*, Preface iii). In a short article, which was probably written around 1915 he writes: "If our language were logically more perfect, we would perhaps have no further need for logic, or we might read it off from the language" ("My Baic Logical Insights", PW 252). Such remarks suggest a conception according to which logical principles derive their validity, or truth, by explicating the essential features which constitute thought and language, and I would like to propose that they are rooted in Frege's conception of logic as a *lingua characterica* which I shall discuss in the next section. (It is plausible, I believe that this whole conception of Frege's of the universality of logic as based on its explicating the essential features of language and thought had influenced Wittgenstein's philosophy of logic in the Tractatus, but I won't discuss this point here.)

Frege's critique of Boole's logic mentioned above suggests not only the requirement that logic should be a universal theory, but also that it should be homogeneous in the sense that the logical principles and notions should be "organically"

integrated in a single theory. In speaking about his notation for generality, which Frege considers "one of the most important components of my ‘conceptual notation'" he adds "[...] in place of the artificial Boolean elaboration, an organic relation between the primary and the secondary propositions is established" (CN, p. 99). And again, in PW, p. 14: "I avoid such a division into two parts, the first dedicated to the relation of concepts (primary propositions), the second to the relation of judgements (secondary propositions) and give a homogeneous presentation of the lot. In Boole the two parts run alongside one another, so that one is like the mirror image of the other, but for that very reason stands in no organic relation to it". In Frege’s logic, in contrast to Boole’s, all logical notions are integrated in one theory; they can "mix" together, get nested and combined in various ways, and do not divide in to separate classes.

This does not mean, of course, that the use of these notions is not subjected to strict rules, but these rules determine both the syntax and the semantics of the logical notions in a way that is fixed in advance, independently of prior classification of the contexts in which they occur.

It is not easy to explain what this notion of the homogeneity of logic amounts to, without entering into detailed technicalities. In general it says that rules are set for forming well formed sentences by predicates and names, irrespective of their specific contents, and that logical operations can be combined and applied in forming predicates and in combining simple sentences to more complex ones, again, irrespective of their specific contents. To see its force let me just give two examples of theories which would not be homogeneous in this sense. The first is of course the very theory Frege criticizes: Boole's logic with its distinction of primary and secondary propositions, which forms, in fact, two different theories. A somewhat less radical example would be a modal logic which would somehow "ban" quantifying into modal propositional contexts. Here again the modal propositions would form a sub-group of propositions that would not integrate with the rest of the theory in a homogeneous way. I suspect that Frege would have objected to such a theory also.

The third point I mentioned in relation to this critique of Boole’s theory is the requirement of explanatory power. Frege’s complaints about the *ad hoc* character of Boole’s laws and the unnatural character of his reduction of secondary to primary propositions amounts basically to the view that a logical theory should be an explanatory one. It must not only determine logical relations, logical validity and truth in an adequate

way; it must also do it in an explanatory way. We should not only know that such and such inferences are valid, etc., but also understand why. This point brings us straight into the next topic, which, perhaps, lies at the bottom of all the previous ones – the conception of logic as an ideal language.

(II) Since *Grundlagen* of 1884, with its emphasis on the "context principle", Frege regarded the entire proposition as the primary bearer of meaning. The meanings of sub-sentential expressions (names, predicates etc.) consist in their contribution to the meaning of the propositions in which they occur. In many of his subsequent writings Frege presents this as a distinctive mark of his approach. For example in "Notes for Ludwig Darmstaedter" of 1919 Frege writes:

> What is distinctive of my conception is that I begin by giving pride of place to the content of the word 'true'... So I do not begin with concepts and put them together to form a thought or judgment; I come by the parts of a thought by analysing the thought (PW 253/273).

Though not stated in these clear terms as a general doctrine, there are hints of this conception in BS and other early writings. In the Preface to BS Frege mentions as one of its main features the replacement of the notions of subject and predicate by argument and function, and writes: "It is easy to see how regarding a content (*Inhalt*) as a function of an argument leads to the formation of concepts". This admittedly is still not a general doctrine about concepts or meanings in general, but an important insight about concept formation. However, it is natural to understand it as backed by such a general doctrine. For, if a concept can be formed in this way, as what "maps" arguments to a content, one can naturally assume that the content of the whole is primary, and the concept just comes by it as a function for an argument. Moreover, concepts thus attained or formed cannot be different in kind from concepts in general, unless one risks losing a grip on the general notion of a concept.

The primary role of the whole propositional content is also indicated by the fact that, after a short remark on his notation for variables and constants, Frege begins BS by explaining the notions of judgement and propositional content, which is called there (quite misleadingly) "conceptual content" (*begrifflicher Inhalt*). This is defined (in §3) in terms of implication relations (what it, perhaps with other premises, implies, and implied by); So, it is obviously propositional – the content of a whole proposition. The same is true of his explanation in the long note *** in "Boole's Logical Calculus" of 1880, PW

10-12). This becomes almost an explicit doctrine in the sequel when he states: "I start out from judgements and their contents, and not from concepts" (ibid. 16) and in point (3) of the conclusion when he talks of "construing judgements as prior to concept formation" (PW 46).

Lingua Characterica and *Calculus Ratiocinator*

Frege had repeatedly emphasized that the comparison between his BS, and Boole's, (or Schroeder's) logic may be misleading because their aims were different: Boole aimed at a *calculus ratiocinator* - a technique for solving inferential problems, while he, Frege, wanted to construct a *lingua characterica*.

> I wished to express content through written symbols […] not a mere *calculus ratiocinator*, but a *lingua characterica* […]. In doing so, however, I recognize that deductive calculus is a necessary part of a conceptual notation (CN 90-91).[6]

In another paper he explains that his aim is the rendering of content in a logical way – that is his idea of *lingua characterica* or conceptual notation. It has two interrelated faces:

> The arithmetic language of formulas lacks expressions for logical connections; and, therefore, it does not merit the name of conceptual notation in the full sense (CN, p. 88).
>
> I would demand the following from a true conceptual notation: it must have simple modes of expression for the logical relations which, limited to the necessary […] must be suitable for combining most intimately with content (CN, p. 91).

Boole, as Frege says, developed "an abstract logic in algebraic formulas", while he, Frege, "wanted to express content through written symbols in a more precise and perspicuous way than is possible with words" (CN, 90-91). Occasionally Frege is even

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[6] At the period prior to his distinction between sense (*sinn*) and reference (*Bedeutung),* Which he drew systematically since "Function and Concept" of 1891, Frege spoke of the content (*Inhalt)* of a sentence. On many occasions this seems to be what he would have later called the thought (*Gedanke)* expressed by the sentence, though on some occasions such a reading may be misleading. Frege simply lacked, then, the conceptual distinction involved, and not only the terminology for it.

rasher as when he says that Boole "fabricated problems in order to solve them with his formulas" (CN, p. 90).

It seems clear that Frege attached the utmost importance to this difference; it is also clear that, as far as logic is concerned, he held quite a low opinion of devising a "mere" calculus. Frege seems to attack not only this or that aspect of a particular logical technique (be it that of the Aristotelian syllogism, or Boole's logic, or whatever), but the very idea of such a technique – of a *calculus ratiocinator* – unless it is combined with and, indeed, forms part of a logical language suitable for the expression of real content – of a *lingua characterica*.

Frege's point may be put here in a broader historical perspective by the following (very schematic) presentation. The task of logic was traditionally conceived as consisting of three main enterprises:

1. Categorization – a systematic classification of structures of sentences and analysis of their constituents;

2. Definitions – or, more generally, a study of concept formation;

3. Derivations – a systematic presentation of valid inference-rules, "rules of proof", etc.

These, though never treated as totally independent of each other, were in fact studied in separate fields of inquiry, and there never was any clear and explicit theoretical explanation of their interrelations.

We may continue this schematic exposition and tie it to Frege's formulations by saying that 1 and 2 above consisted of what a *lingua characterica* was concerned with, while 3 comprised of what a *calculus ratiocinator* was supposed to accomplish. Frege's point may be put now as a call for the unification of all the three enterprises: the same logical functions or operations (the same logical notions) should be common to all the three. In other words, the same set of logical notions should be operative in spelling out the kinds and structures of sentences, the definitions or other means of concept formation, and the rules of inference and logical laws governing implications between sentences. It is this kind of unification that Frege appeals to in his repeated saying that "logical connections should be naturally integrated with the expression of content". Such a structure would form a real *lingua characterica* and Frege's *Begriffsschrift* was aimed at

accomplishing (and to a large extent has accomplished) exactly that; it is, as he says, his "fresh approach to the Leibnizean idea of a *lingua characterica* (PW 10).[7]

In explaining the advantages of his *Begriffsschrift* over natural language, Frege says: "Language is not governed by logical laws in such a way that mere adherence to grammar would guarantee the formal correctness of thought processes" (CN, 84-85). Adherence to grammar has to do with the inner structure of sentences; formal correctness ot thought processes – with derivations and implications. And later he complains that "In this process (of testing validity), since insufficient security lies in the nature of the word-language itself, the laws of logic are applied externally like a plumb-line" (CN, p. 85). What we need, instead, is "a system of symbols from which every ambiguity is banned, which has a strict logical form from which the content cannot escape" (CN, p. 86).

Here, it seems to me, we have an embryonic formulation of the idea of logic as a theory of reference (of meaning). Logical form should provide a clear and unambiguous presentation of the content of a sentence and at the same time it must be couched in the same terms by which the "logic" or the rules of valid inferences of the language are expressed. As Frege expressed it in another passage quoted above, in an ideal language we should be able to "read off" the logic from the grammatical structures of the sentences.

Analogous remarks apply to Frege's treatment of definitions and concept-formation.

> The real importance of definition lies in its logical construction out of primitive elements […]. The insight it permits into the logical structure is not only valuable in itself, but is also a condition for insight into the logical linkage of truths (*Grundlagen der Geometrie*, p. 302).

On many occasions Frege remarks, in praise of his definitions and in criticising others, that the real merit of a definition proves itself in the use one makes of it in proofs and derivations. A sure indication of the emptiness of a definition is the fact that it is not used in proofs, or in derivations. In light of our previous remarks what that means is that definitions should be formulated in terms of logical notions that yield structures on which the rules of inferences are defined. Classical and powerful examples of these are provided by the BS definitions of the hereditary and the "follower in a sequence". These

[7] "Fresh approach" is perhaps an understatement, and in a late piece Frege in fact regrets using the term for its associations with the ideas of "Leibniz and his followers" ("Notes for Ludwig Darmstaedter", PW 253/273).

definitions, which are couched entirely in terms of the logical apparatus of the BS, are essential to the the proofs of the principle of induction and related theorems of that work.

This view may also explain one of the main points Frege often makes in characterizing his way in logic, which we have already quoted:

> What is distinctive about my conception of logic is that […] I do not begin with concepts and put them together to form a thought or a judgement; I come by the parts of a thought by analyzing the thought. This marks off my concept-script from the similar inventions of Leibniz and his successors ("Notes for L. Darmstaedter", PW 253).

This is so because definition and concept formation in general are, for Frege, constrained and even determined, by his organic or integrated conception of the logical structure of sentences and the theory of inference. This is the kernel of his conception of logic.

There results, in sum, a particularly impressive conception of logical theory which integrates, in a coherent way, a theory of the structure of sentences, a theory of definition and concept formation, and a theory of inference and proof. This integral conception, I suggest, is the main characteristic of Frege's novel conception of logic as *lingua characterica*, and it is what gives its (normative) deductive part its explanatory value.

Chapter 3: SENSE AND OBJECTIVITY IN FREGE'S LOGIC

The essentials of Frege's revolutionary logic appeared in his *Begriffsschrift* (BS, 1879). Important aspects of its philosophical basis, and its significance for the foundations of mathematics, appeared in *The Foundations of Mathematics* (FA, 1884). Six years later, at the beginning of the 1890s, Frege published three articles that mark significant changes in his conception: "Function and Concept" (FC, 1891), "On Sense and Reference" (SR, 1892) and "Concept and Object" (1892). Notable among these changes are: (a) The systematic distinction between the sense and the reference of expressions as two separate ingredients of their meaning. (b) The extension and generalization of the notion of function to include the conception of concepts and relations as functions to truth-values, and the corresponding conception of the two truth-values as objects.[1] These changes were immediately incorporated in the mature, authoritative exposition of his logic in his *magnum opus: Basic Laws of Arithmetic* (BL), whose first volume appeared in 1893.

What is the role of the notion of sense and of the distinction between sense and reference in Frege's **logic**? Is there a systematic connection between the two points (a) and (b) mentioned above, so that their being incorporated together in Frege's mature logic is not accidental? These questions, I believe, are central to understanding Frege's mature conception.

In the sequel, after presenting the problem in a sharper way (A), I shall sketch what seems to me the general direction of an answer (B-C), and then add further clarifications of related issues (E-D). In three sentences the general direction is this: Logic, in its wide sense, is, for Frege, the science of justification and objectivity. These are correlative notions: the objective is what is justifiable, and justification requires objective standards. The role of the notion of sense in this enterprise is in establishing the objectivity of the basic truths of a domain (including logic itself), which is accomplished by presenting these truths as expressing features of the ways in which the objects of the domain are given to us. This appeal to objects and their modes of presentation gives a particular realistic turn to Frege's notion of objectivity:

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[1] The extension of the notion of a function was already a major theme in BS(see BS section 9). But there it was introduced as an expression, and was perhaps flawed by a confusion of sign and thing signified, or expression and content expressed. Moreover, since Frege thought there in terms of a general undifferentiated notion of content, and he did not clearly distinguish a function from its values, essential features of his conception remained unclear. I shall not go into the details here.

One face of it connects it to justification; another, to objects and their modes of being given to us.[2]

(A) Two Characterizations of Sense

1. The Core Idea – Sense as a Mode of Being Given

Frege's notion of sense is usually presented as stemming from epistemological considerations, as carrying the "cognitive value" or informativeness of expressions and sentences. Various formulations of Frege's provide support for such a view, notably the famous paragraph at the beginning of SR, where Frege argues for the need to distinguish the sense from the reference of expressions. Apart from some general principles that govern these notions, the reference of a term is explained as what it denotes in its use in simple sentences, and what these sentences are about.[3] The sense of a term is introduced, on this conception, as the mode or way in which its reference is given to us. Thus "The morning star is the evening star" is true in that the two names have the same reference, and it is informative in that they have or express different senses in which their "cognitive value" is contained; the sentence as a whole expresses the thought to the effect that these two senses belong to the same reference.[4]

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[2] Sense, in Frege's philosophy, appears to belong to logic most clearly in definitions and in his theory of definition. I shall not go into these issues here. In its strict sense a definition for Frege is a stipulation of synonymy, as it stipulates identity of sense between definiendum and definiens. The definiendum in fact gets its meaning from the definiens through this stipulation. Clearly in such a conception the distinction between reference and some notion of sense is mandatory. Frege also recognized another, less strict notion of definition in which the definiendum is a term in current use, already endowed with meaning. Here again questions of adequacy require appeal to some notion of sense. What the roles of these kinds of definition in Frege's logic are, whether these notions are the same, and what are the relationships between them are serious problems I shall not discuss here. I elaborated on some of them in "Conceptual Analysis and Analitic Definitions", here chapter 9.

[3] In these governing principles lies the great novelty of Frege's notion of reference; I shall not expand on it here. On the significance of this appeal to the notion of about here, and for references, see chapter 7 of my book (1996).

[4] Important ingredients of the core idea already appear earlier, though without the systematic terminology. See for instance BS, section 8 (and chapter 4 here); FA, section 62. Compare also my book (1996) pp. 44, 54, 111.

As I said before, this is a prevalent conception of Frege's notion of sense, which finds its clearest formulation in Frege's own writings primarily in SR, and throughout his writings since then (e.g. T in FR, 333/65-6). Following the terminology I used in my book (1996, p. 7) I call this notion of sense "the core idea" of sense. There is something "local" and lexical about this notion of sense: One begins with the senses of individual simple names, and moves on "from the bottom up" to more complicated ones. It is quite late in the article (32/62) that thoughts are presented as the senses of complete sentences (where presumably the notion of sense is taken as already understood). And nowhere in this article does there occur the crucial idea (central in other writings) that the sense of an expression is a constituent of a thought – the particular contribution the expression makes to expressing this thought.[5]

Being epistemically governed, this notion of sense is also individuated in epistemic terms. Frege often proposes or assumes that two senses are the same if and only if whenever one knows them one knows they are the same. Put in different terms, the criterion says that two expressions express the same sense if and only if it is impossible to fully understand them both, yet fail to know that they express the same sense.[6]

This account may be correct as far as it goes, but it does not go far enough. For Frege introduces and uses the notion of sense in his distinctly logical writings (e.g. FC, BL). One can naturally wonder about the role this notion of sense plays there, in Frege's logical doctrines, even if one grants the above account of its role in Frege's epistemology, and in his account of various features (such as cognitive value) of natural language sentences and expressions. Logic, as it is often conceived, is concerned with a clear and systematic presentation of deductions and proofs, and so it

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5 This article is unique among Frege's writings in being directly concerned solely with natural language. This is important because, on the one hand, it clearly shows that Frege intended to apply his notions of sense and reference to natural language sentences and expressions, while on the other, it may explain the particular way in which the introduction of the notion of sense here diverges from the way it is presented in later writings of a more logical orientation.

6 See, for instance, "A Brief Survey...", PW p. 197. The principle is also assumed (though not stated) by Frege in T (65/25; cf. my book p. 70). It should be noted that there are other criteria, which Frege proposes and uses for sameness of senses and thoughts. See, for instance PW 140, letter to Husserl of 9 Dec. 1906, PMC 70. For a broader discussion cf. my book (1996) pp. 214-217.

was generally conceived by Frege as well.[7] Sense, it may be claimed, does not belong here, even granted its importance in accounting for other, non-logical, features of sentences (in a natural language as well as a logical one).

To highlight the point it is quite typical that, although modern systems of logic derive their essentials from Frege's logic (the generalized function-argument conception, the conception of sentential logic as the logic of "truth functions", i.e. functions over truth-values, the basics of quantification theory, etc.), the most distinguished exception is Frege's notion of sense, which is hardly mentioned in most courses and texts of classical modern logic. Logicians do not seem to need this notion; many of them hardly know anything about it, and many of those who do, who tend to be more philosophically oriented, explicitly and doctrinairely reject it. All this may strengthen the suspicion that, in trying to incorporate the notion of sense into his logic, Frege slipped or was appealing to a different notion of sense.[8]

2. Sense as a Constituent of Thought

There is, indeed, another conception, or perhaps merely a different emphasis in the conception of sense, which is found mainly in Frege's later writings, and is dominant in his logical works. In this latter conception, thought is the primary notion, where

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[7] See for instance "Logic", PW pp. 3, 4; "17 Key Sentences...", PW p. 175. This conception of logic is related to Frege's repeated claim that truth, as distinct from the recognition of truth, is the subject-matter of logic (PW 128/139; the beginning of T)

[8] With all its reliance on Frege's ideas, there are some respects in which modern logic took a course different from his, notably in its reliance on set theory and model theory. It may therefore appear that the role played by the notion of sense in Frege's logic is somewhat analogical to the role of the notion of a model in post-Tarskian logic. In some extensions of standard model theory in modal logic this idea has found explicit formulations, e.g. by Hintikka, who has explicitly suggested explicating Frege's notion of sense in terms of the extensions of terms in various possible worlds. See e.g. his 1969, p. 105. I believe that this proposal is mistaken and misses some essential features of Frege's notion of sense (e.g. its particular perspectival cognitive character, and its intentionalistic nature (see my book 1996, ch. 1), but it may support the general point made above, that in Frege's logic, the notion of sense played a similar role to that played by model theoretic notions in later developments of logic.

senses are conceived as parts of thoughts (their "building blocks"). The sense of a (declarative) sentence is identified with the thought it expresses and the senses of its constituent expressions are presented as their contribution to that thought, or parts of it. This is the dominant conception of sense in Frege's later writings. It occurs as early as FC (13-14/29), but gets its conspicuous expression e.g. in section 32 of BL:

> The names, whether simple or themselves composite, of which the name of a truth-value consists, contribute to the expression of a thought, and this contribution of the individual [component] is its sense.

Again, following the terminology I used in my book (1996, p. 7-8), I shall refer to it as the thought-constituent notion of sense. This conception is obviously connected with the context-principle, and seems to be significantly different from the "local-lexical" conception of the "core idea". The difference is perhaps most conspicuous if one considers a thought (as many believe that Frege did) as a Platonic entity, existing in itself, independently of human minds (though perhaps, as Frege says in FA, not of The Mind), and being true or false independently of our cognitive limitations. This notion of thought has not so much to do with knowledge and modes of presentation, as with logic and logical relations: A thought is what is true or false, and what stands in logical relations of deducibility, contradiction and so forth, with other thoughts. It may seem that unlike the core idea, senses of sub-sentential expressions are not conceived of as modes of presentation of their references, carrying their "cognitive value", but as constituents of thoughts, whose very being and individuation depends entirely on that of the thoughts containing them and their logical structure.

It is, moreover, difficult to see how sense can be conceived as mode of presentation when the notion of thought is regarded as the primary notion of sense and thus the basis of any other kind of sense. For it is difficult to see what the notion of the mode of presentation of a truth value could amount to, even apart from the grave difficulties in conceiving truth values, regarded as the references of sentences, as objects. As I argued in my book (1996), Frege's notion of sense is primarily intentionalistic, and the idea of mode of presentation is vital to it. This idea is intuitively well understood (or at least relatively so) with regard to objects – primarily concrete "ordinary" objects, and with some refinements and adjustments, abstract objects; it begins, however, to creak and squeak when applied to concepts and functions; it seems completely ad hoc and unintelligible with regard to truth-values. It therefore appears that we are faced here with two unequivalent characterizations of sense, and one may wonder whether the primacy of thoughts and the conception of sense as thought-constituent can be naturally reconciled with the core idea of sense.

Looking back at our opening questions about the role of the notion of sense in Frege's logic, it appears now that the notion of sense relevant to logic is not the

epistemic notion of the core idea (a mode of being given), but the later idea of a thought-constituent.[9] Logic, as mentioned before, is concerned with a clear and systematic exposition of inferences or in justifying and establishing truths on the basis of other truths. Whatever is relevant to this task is logic's concern. A thought is what is true or false, and in that sense thoughts form the subject matter of logic. Moreover, it is logic (including the study of the structures and properties of logical languages) that determines what a thought is: A thought is that part of the content of a sentence, which is required for a clear and systematic exposition of the logical relations it can enter into, and, in particular, what is required for establishing and justifying the truth of a statement on the basis of other truths.[10]

Frege's repeated objection to the conception of logic as an abstract, purely formal calculus that can be interpreted in various models is a facet of his insistence on conceiving logical formulae as "full blooded" statements, expressing thoughts which are true or false. [11] This is an essential point in his conception of logic, and it is a point that places the notion of sense immediately into the very center of logic. Logic for Frege is concerned primarily with thoughts and it is what determines the parameters that make a thought what it is.[12]

These points about the difference between the core-idea and the thought-constituent conceptions of sense may appear to threaten the coherence of Frege's notion of sense. It is not clear, however, that they amount to more than a difference in emphasis. I have mentioned before that the difference is most conspicuous on a Platonic conception of thoughts (and senses). It may thus appear that the difference may be diminished if this Platonic conception is rejected, as it is arguable that it suggests a misleading picture of Frege's conception of thought, and that a thought, as well as the conception of the structure of a thought and the contribution its constituents make to it, are themselves epistemic notions, or epistemically constrained. This is the direction taken in my book (1996), where the connection

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9 See the proviso about definitions in note 1.

10 This is central to both the "Logic" of the 1880s and to the later "Logic" of the 1890s (as well as many other writings), both written probably as parts of a general textbook on logic. See PW pp. 1-8; 126-151. The notion of thought here stems from "conceptual content" of BS, which is characterized in similar terms (see, BSsection 3).

11 See Bar-Elli (1985), here chapter 2.

12 See Frege's PW (1979) pp. 197-8; compare also C. Diamond, *The Realistic Spirit* (1991), pp. 115-144, especially 117-120.

between these two faces of sense is explained along these lines (see, e.g. p. 15). Yet recovering the coherence of Frege's notion of sense in this way may be gained at the price of rendering it in its entirety irrelevant to logic. For it is doubtful whether such an epistemic notion can carry the burden of an objective conception of thought as what is true or false (under bivalence) and as what stands in logical relations, in the strong classical sense. It is also for this reason that the difference seems significant enough to deserve further attention, and that the relations between the two notions (or two aspects of the notion of sense) require careful study.

(B) Sense and Justification – The Coherence of Frege's Notion of Sense

The previous considerations seem to threaten to tear Frege's notion of sense apart, and to show that the notion of sense relevant to logic is the thought-constituent notion. This, I believe, is too hasty a conclusion. I shall try to show this by pointing out the role of the core idea in logic and in establishing the objectivity of logic. Being thus placed at the center of logic, the core idea is seen as correlative and complimentary to the thought-constituent conception of sense. With reference to our opening question about the role of the notion of sense in Frege's logic, we shall thus see that it is precisely its role in logic, conceived as the science of objectivity, which restores the coherence of the Fregean notion of sense.

The key to understanding this is the role of the notion of sense in establishing the objectivity of a domain of thoughts. The criterial sign of objectivity, for Frege, is justification or justifiability: something is objective insofar as it is justifiable or as statements about it are. The first of the three principles in the Introduction to FA is the demand to distinguish between the logical and the psychological, the objective and the subjective. These are parallel distinctions. It appears therefore that the objective is the logical. And logic, as Frege makes clear on numerous occasions, has to do with the justification of propositions. (See the Preface to B; FA section 3; PW p. 3; PW 147.) The main tenor of FA is to establish the objectivity (or "objective factuality", as Frege sometimes says) of arithmetic by clarifying the grounds or justification of arithmetical propositions. Logic, objectivity and justification form an inseparable triad for Frege. This, however, raises a question, for in its simple sense (and so also in Frege) what is objective is what is "out there" in the world, or what concerns objects that are out there in the world, and which are accessible and can be examined by different people, from different perspectives, etc. What, then, is the relationship between these two aspects of the objective: the logical aspect, on the one hand, and being concerned with and based on objects in the world, on the other? In trying to answer this, I believe, we must appeal to the third element of the triad – the notion of justification. What is objective is only what is justifiable or what is used in a

justification; in short, what is in the justification-space. Logic is not only itself objective in this sense, but is constitutive of objectivity. It is what sets the standards for justification and objectivity, what constitutes the justification-space.

The main task Frege set up for himself was to establish the objectivity of various domains – particularly mathematics (arithmetic). For him this means presenting statements in this domain as justified or at least justifiable. Logic was the paradigm and primary means of such justification. Deductions and proofs are chains of justifications of some truths on the basis of others. In this sense, logic, being the paradigm of justification is also the heart of objectivity. It is not only a paradigm of objectivity, but it is what sets the standards of objectivity, and is thus constitutive of the very notion of objectivity. It is obvious that on this conception both logic and objectivity have to do mainly with the notion of sense as thought-constituent. The core-idea notion of sense seems to be out of the picture.

But, and this is the main point, there is another level, or stage of justification and objectivity, which is not strictly deductive, but can still be considered as logical in the wide sense. Logic in this wide sense is precisely the theory of justification or justifiability.[13] This other level is where we justify the basic truths of a domain not by proving them or deriving them from more fundamental truths (since there aren't any), but by showing them to be clear, justified, or evident by the way "their objects", the objects they are about, are given to us. One can even say that these basic truths are justified by the fact that they express (aspects of) the ways "their" objects are given us, or, in other words, by the modes of presentation or senses of these objects.

Frege's primary example of this was geometry. Geometrical truths (theorems) are objective in being justifiable. They are justified, through logical proofs, on the basis of other geometrical truths, and ultimately by the basic truths, or axioms of geometry. But what about these axioms themselves? Obviously, they cannot be derived logically from more basic truths. Should we say they are not justifiable, and therefore not objective? Certainly not. This would ruin the objectivity of the whole edifice built on the basis of these axioms (cf. FA section 26). It is here that this other form of justification is used. The axioms are justified on the basis of the ways their objects, the objects of geometry, are given to us: "Everything geometrical must be

[13] See, for instance, "Logic", PW p. 3; "17 Key Sentences..." ibid. 175. Cf. also my book (1996) pp. 40-46, and in more detail Bar-Elli (2006), here chapter 7.

given originally in intuition" (FA, p. 75). The axioms, one may say, express at least some basic aspects of these modes of presentation of the geometrical objects.[14]

Frege sets himself the task of constructing something similar for arithmetic, thus establishing solid foundations for mathematics. The defects he found in arithmetic were not only that many arithmetical proofs were unclear or obviously faulty, but also that the whole science of arithmetic lacked an objective basis. To remedy the former fault it was necessary, according to Frege, to present arithmetical thoughts in a systematic logical language that would render their proofs transparent and detectable. That was obviously the task of logic, or of constructing a logical language. In principle this was accomplished by the logical language of BS (1879). But the latter deficiency was no less severe, and called for no lesser a task: the task, namely, of presenting the objects of arithmetic, the objects the axioms are concerned with (namely, numbers), in such a way that the axioms themselves will be justified.

Frege's logicism – the program of presenting arithmetic as logic – was designed to solve both problems. Expressing arithmetical truths in the language of the *Begriffsschrift* enabled him to present their proofs completely and systematically. But it served a further and, in a way, more basic aim. Being convinced that there is no other way in which numbers can be construed that can justify their axioms, Frege thought that the only way to achieve this goal was to construe them as logical objects. This, together with a logical rendering of other arithmetical notions, would enable us to present and justify the arithmetical axioms as logical truths. Again, the crucial step here is the double move of regarding the axioms, the basic truths of arithmetic, as being about objects of a certain kind (numbers), and regarding the ways these objects are given us as justifying these axioms, thus establishing the objectivity of the whole edifice of arithmetic. This double move may explain Frege's persistent view that establishing the objectivity of arithmetic requires an explication of the nature of numbers and their way of being given to us. This was not just a sort of a Socratic wondering about essences; it was rather a requirement implied by the conception of objectivity under discussion, and by the task of establishing the objectivity of arithmetic. It is evident (as, e.g., the structure of FA makes manifest) that had Frege been satisfied with other ways of establishing the objectivity of arithmetic and explicating the nature of numbers, he would have seen no need and no point in reducing arithmetic to logic. But, of course, for most of his life Frege was convinced that no non-logical account was possible, and was thus "forced" into his logicistic

[14] Scholars have debated the question of whether this way of being given is basically Kantian spatial intuition, and whether Frege can be regarded as a Kantian in this respect. I shall not discuss this here. Cf. Dummett, IF (1981b), pp. 463-470.

project. Frege's logicism is, from this point of view, not only a reductivist program and a technical achievement, but also a philosophical discovery: it is the discovery that one can present the basic truths of arithmetic as being about objects whose mode of presentation to us as logical objects justifies these truths. Let me elaborate a bit.

(C) Logical Objects – Frege's Fundamental Principle

The idea of logical objects is a notorious one. It was not part of the lore of logical tradition, which makes it even more puzzling why Frege held so firmly to it.[15] Moreover, the inconsistency of the fundamental axiom (V) in BL, which concerns logical objects (extensions), gave a fatal blow to Frege's "life-project", according to his own admission. Why then was he so persistent (some would say obsessed) about logical objects? I suggest that the answer is important for understanding not only Frege's conception of logic and mathematics, but also his conception of objectivity and sense, and their interrelationships. I offer a partial answer to this puzzle by suggesting two general points and a more specific one:

(a) The idea that any truth or any meaningful statement must be about something, primarily about objects.[16]

(b) The idea that the basic truths of a domain are objective in that they are justifiable, and that the justification of such truths or axioms can be attained in terms of the ways the objects they are about are given to us. (This is the main thesis argued for here).

There must of course be objects, in order for there to be ways in which they are given to us. Hence, these two principles imply that the objectivity of logic as a system

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15 "Object-producing principles" were at the basis of Frege's philosophy as early as FA (1884). There Frege adopted a version of the "Hume-Principle", which says, roughly, that the object: the number of Fs is the same as the number of Gs iff the concepts F and G are equinumerous (equipotent). George Boolos saw this as a principal reason for regarding Hume's principle as non-logical (1990, 261-77).

16 Frege held this doctrine throughout his career. See e.g., the early "Dialogue with Punjer on Existence" (PW 53-60), and the late "Numbers and Arithmetic", where he writes: "I, for my part, never had doubt that numerals must designate something in arithmetic, if such a discipline exists at all [...] We do after all make statements of number. In that case what are they used to make an assertion about?..." (PW 275). The *locus classicus* of this view is FA section 46, and ch. IV. For a detailed discussion of this point see ch. 7 of Bar-Elli (1996).

of truths based on logical axioms rests on the existence of logical objects, whose ways of being given to us justify the axioms (even if only partially).

(c) Now the specific point: The specific reduction of arithmetic to logic suggested by Frege required appeal to classes (in one version or another), and the immediate Fregean question was how these classes are given to us. Since this is conceived as a reduction to logic, then, if the above position is valid, these classes must be given to us as logical objects. Giving up appeal to how these objects (classes) are given to us as logical objects (their senses) means giving up an account of the objectivity of arithmetic and parts of logic. This was a price Frege was unwilling to pay, and it is hard to blame him for this.

The last point involves some technical questions concerning the exact nature of the reduction involved and the set theory assumed. I shall not get into these issues here, but will confine myself to the essential general point: that on Frege's conception the objectivity of logic and its basic truths relies on there being logical objects and on the ways they are given to us. In order to see that, it is worthwhile to recall some features of the way in which logical objects are introduced (or discovered) in Frege's mature logical system. As early as 1891 (cf. FC, p. 9-10/26) Frege presented the basic idea of what was to become the fifth axiom of BL. He explains and defends a move from a general equivalence of the form (x)(Fx↔Gx), what he calls there an "equality holding generally between values of functions", to an identity between objects – the "courses (or ranges) of values" of these concepts: x^Fx=x^Gx.

The move from the former to the latter – from the equivalence to the identity – is, according to Frege, irresistible and reliant on a fundamental and indemonstrable logical principle. It is this relationship that expresses the way the logical objects, the courses of values of functions (including extensions of concepts), are given to us. This then is the way in which we get to the logical objects – objects which are given to us by way of logic. Ignoring irrelevant subtleties, we may regard these objects, the courses of values, as classes. The above, then, is the move by which we get classes as logical objects with which the relevant part of logic – predicate logic – is concerned. They are given to us in a particular way which is expressed by that elementary move from general equivalence to identity. And it is this mode of being given that is purported to justify the relevant axiom (axiom V), which says roughly x^Fx = x^Gx ↔ (x)(Fx↔Gx), as the axiom expresses an essential aspect of this mode.

Of course, I do not intend to defend here an axiom that has been proved inconsistent. This particular account of the way the logical objects are given to us has

been proved a failure.[17] But this does not mean that the basic position on the relation between objectivity and the ways objects are given to us – a position from which this account emerged – is faulty. It is this position, in its Fregean version, that I am trying to explain. Even if Frege's particular account was a failure, it is important to understand an account of what it was intended to be. It may be relevant to recall that even when, towards the end of his life, Frege gave up the logicistic approach to arithmetic, he still kept unquestioned the view argued for here: that accounting for the objectivity of arithmetic requires an explication of the nature of arithmetical objects and how they are given to us, which he then suggested should be construed on the basis of geometry (cf. "Number and Arithmetic" (PW 275-7), and "A New Attempt..." (PW 278-281)). Moreover, his qualms about axiom V of BL notwithstanding, he never questioned the very idea of logical objects and the need to appeal to them in accounting for the objectivity of logic. He kept talking of the "logical source of knowledge", which requires logical objects (ibid.) and there is no sign that he ever questioned his view that truth and falsity are logical objects.[18]

It may be instructive to consider a passage Frege wrote to Russell (after learning of the contradiction) on 28.7.1902:

> "I myself was long reluctant to recognize ranges of values and hence classes; but I saw no other possibility of placing arithmetic on a logical foundation. But the question is, How do we apprehend logical objects? And I have found no other answer to it than this, We apprehend them as extensions of concepts, or more generally, as ranges of values of functions" (PMC 140-1).

Classes or courses of values are not the only logical objects in Frege's system. There are also two other important logical objects – the two truth-values – the True and the False. These two, as we shall see, are obtained by a similar move from equivalence to identity, although in this case the original move, in SR, is not made by Frege explicitly. In SR truth-values are introduced as the references of sentences

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17 Frege, as has been often noted, had reservations about the fifth axiom of BL right from the beginning, even before learning of the contradiction.

18 In IF (1981b) p. 464 Dummett writes that for the late Frege (in "Sources of Knowledge..." of 1924/5) "there are no objects given by logic alone". I don't think this is correct. Frege indeed denies there that sets are objects, and speaks of the dangerous ways in which language can mislead us to postulate objects where there are none. He even speculates that number-words are not names and do not designate objects. But he does not say that there are no logical objects.

by a somewhat strange argument to the effect that if we are concerned with the references of proper names (or sub-sentential expressions), this cannot be because of the thought expressed by the sentence (for which they are irrelevant), but only for its truth-value, which must then be regarded as the reference of the sentence (33/63). Frege then proceeds to say in a uniquely uncritical move:

> "Every declarative sentence concerned with the reference of its words is therefore to be regarded as a proper name, and its reference, if it has one, is either the True or the False. These two objects are recognized, if only implicitly, by everybody who judges something to be true..." (ibid.)

Even if we are ready to accept this strange version of the slingshot argument – ascribing a reference to a sentence – Frege's further move of regarding this reference as an object and the sentence as its proper name might still appear singularly strange and unmotivated. But the idea is quite similar to the one we have seen above with regard to the introduction of value-ranges as objects: an irresistible move from equivalence to identity, from 'P iff Q' to 'p=q', (the capitals stand for sentences, combined into a complex sentence by a sentential operator, and the small letters stand for the appropriate names of truth values). Putting the point in a way more similar to Frege's principle, we might introduce a special operator '*' carrying propositions (or thoughts or the contents of statements, whatever they are) into objects. The transition would then be from 'P↔Q' to '*P = *Q'. We may accordingly formulate a principle, somewhat analogous to axiom V, which would on the one hand express, and on the other be justified by, this elementary logical transition: (*) *P = *Q ↔ (P↔Q). Again, this irresistible move from equivalence to identity expresses a feature of the way we get to these objects – logical objects that are given to us by way of logic.[19]

Here, within the domain of truth-functional logic the equivalence plays a somewhat similar role to that of the general equivalence in the predicate logic. Both express individuation conditions of the basic units of the logical domains concerned: truth and falsity here, functions and their range-values there. In both cases Frege assumed (once explicitly, once implicitly) an elementary and irresistible transition from the equivalence in question to identity between objects. This is the crucial step in

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[19] In my article, "Identity and the Formation of the Notion of Object", *Erkenntnis* 17 (1982), pp. 229-48, I proposed that the ontology of a theory is determined by the "identity stipulation", which identifies a general equivalence relation (indiscernibility defined over all the descriptive predicates of the language) with identity. I was not aware then of the affinity between this idea and the basic Fregean idea I am trying to present here.

which he introduced (or discovered) logical objects as the basis of the objectivity of logic.[20] These objects are the basis of the objectivity of logic in the sense that their modes of presentation, expressed by the elementary transitions involved, justify the basic truths of the said logical domains: "The truth of a logical law is immediately evident from itself, from the sense of its expression" (CP 405). This, according to Frege, is the only way in which such basic truths (axioms) can be justified, hence they form the basis of their objectivity. Senses, according to the core idea, are such modes of presentation, which means that they stand at the very basis of the objectivity of the logical axioms.[21]

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[20] Frege says that axioms and basic logical truths aare "general" (e.g. FA section 3). This means, I suppose, that no particular objects (or functions) are referred to. It seems to me that this does not conflict with my emphasizing objects and their ways of being given as an epistemic ground of the axioms, for I am obviously talking about kinds of objects (truth-values, courses of values of functions) and not particular objects. For example, the propositional axiom a⊃(b⊃à) is, in Frege's mature conception, a generalization over truth-values. It does not refer to a particular truth-value or object. On the conception I propose here it expresses a feature of the ways truth-values are given to us, wherein its justification lies.

[21] The above formulations are quite general and should be qualified in various ways. One of the most important is this: It could be argued that the logical transition mentioned does not express the sense of, e.g., truth-values, for the sense of a truth-value is a particular thought, and the sense of, e.g., a class is certainly more complex than the said elementary logical transition, and contains an element that relates to the sense of the function determining that class. This is true when the sense of a particular truth-value or a particular class is concerned. I am discussing more generally the way in which truth-values and classes are given to us. A thought expresses the sense which "belongs" (as Frege says) to its truth-value, but this is subordinate to the mode in which truth-values are given to us as objects in general. Similarly, a particular sense belongs to a class under the mode in which classes are given to us in general. The elementary transitions discussed are intended to express such general features of the mode of presentation of logical objects.

(D) Frege's Principle and Equivalence-Relations

The equivalence involved in Frege's fundamental principle is not exactly an equivalence-relation, but rather a propositional equivalence. Not just any definition in terms of equivalence relation discovers or presents objects in the sense in which logical objects are presented by Frege's principle. One might fix the meaning of identity between such things as spatial directions or income levels in terms of the appropriate equivalence relations. But these would not amount to "discovering" these objects and would not express the ways these "objects" are given to us, similarly to the way I described above with regard to logical objects. Some light may be shed on this if we compare the above with what might seem to be a similar move in FA. It must of course be remembered that in FA Frege held neither the systematic distinction between sense and reference, nor the doctrine of truth-values as the references of sentences. Moreover, the correlative doctrine of sentences as proper names of objects (the truth-values) seems to conflict with the emphatic conception in FA of sentences as being radically different from names, and of their having a unique and primary character.

In a famous section Frege prefaces to his proposed definition of number in FA (#64) he says that we can introduce directions (as objects) by stipulating that the directions of two lines are identical iff the lines are parallel (D(a)=D(b) ↔ a//b). This again is a sort of transition from an equivalence to strict identity. Frege agrees with Kant that lines and parallelism are grounded in our (spatial) intuition, and the above principle expresses (at least an important aspect of) a way in which directions might be given to us as objects.

There is, however, a great difference between this example and the general point we made above. The equivalence we talked about in connection with axiom V and (*) is "propositional equivalence"; from a modern point of view, it is, on the face of it, not a relation at all, but is expressed by a truth-functional connective between sentences. The equivalence on the basis of which Frege introduces directions in FA, in contrast, is an ordinary equivalence-relation, which is a relation between objects.[22] The

[22] The status of the general equivalence relation of "indiscernibility" in the "identity stipulation" in my paper in note 18 above is more complicated. It may appear that it is an ordinary equivalence relation between objects. This however, is not exactly so. The indiscernibility relation is intentionally defined in terms of propositional equivalence with no presumption of an objectual construal of the semantics involved. The whole point of the "identity stipulation" proposed there is to fix the ontology of the language on the basis of a non-objectual basis.

difference I have in mind may seem somewhat non-Fregean in spirit. For in *Begriffsschrift* Frege introduces "identity of content" for "judicable" (i.e. propositional) and non-judicable contents alike. But, besides a notorious confusion of use and mention in that early discussion, the distinction concerned (between propositions and objects) became to be of central importance for Frege in FA. The difference is again blurred by Frege's later view that equivalence (like other propositional operations) is a (first-order) relation of truth-values. I think, though, that the difference holds also for this late view, since even on this view, truth-values are very special objects, and there are reasons to believe that in his mature and late writings Frege continued to adhere to his context principle and the primacy of propositions proclaimed in FA. But I shall not go into this much debated point, for here I am concerned with FA, in which Frege had not yet held this view of truth-values as objects.[23]

The definition of directions in terms of the equivalence relation of parallelism, discussed in FA, is, therefore, not an instance of that transition from (propositional) equivalence to identity of which we spoke above. The problem with such definitions is not only that they are contextual – that what is defined is only the whole identity context of, e.g., D(a)=D(b). It is also the (related) problem that they are definitional stipulations that fix the mode of presentation "vacuously", and do not express an independently existing mode of presentation of the objects concerned. The definition as a whole is therefore not a justifiable truth (though, as Frege himself remarked, it itself may justify a corresponding truth). This is part of what Frege means by saying that one can understand that a//b with no appeal to directions at all. With regard to truth-values, on the other hand, Frege asserts that "these objects are recognized, if only implicitly, by everybody who judges something to be true". The move from

[23] In a letter to Russell of 28.7.1902 Frege, mentioning his view in FA, explains the transition from equivalence-relation to identity as the general principle behind Russell's "definition by abstraction". He then remarks that the difficulties involved with this procedure of definition by abstraction are the same as those of "transforming the generality of an identity into an identity of ranges of values" (PMC, p. 141). The English translation here mistakenly put "not" into the sentence (probably a misprint), as if the difficulties concerned in the two cases are different. The original is: "Die Schwierigkeiten sind hierbei aber dieselben, wie bei der Umsetzung der Allgemeinheit einer Gleichheit in eine Werthverlaufsgleichheit" (p.224).

parallelism to identity of directions is not the sort of irresistible, fundamental transition in terms of which Frege describes his axiom V.

Directions are not (basic) objects on which the objectivity of geometry rests. Accounting for the way in which they are given to us is thus no part of an account of the objectivity of geometry. Directions **can** be introduced into geometry by the above principle. They can be also explicitly defined, on familiar lines, on the basis of the principle. But this is not necessary for the objectivity of geometry. What is required is a conception of the basic geometrical objects – lines and points. These are given to us in a particular way expressed (in part) by the geometrical axioms, thus forming a justification for the axioms themselves.[24] The logical case we discussed above is similar: Here we are looking for the basic objects whose modes of being given to us form the basis of the objectivity of the logical domain. What, then, is the point of the example about directions, which Frege discusses at such length?

In FA Frege did not pretend to have "discovered" that there are arithmetical objects, nor was this his aim. That numbers are objects is a fact he was convinced of prior to the logical analysis of their nature. If numbers in arithmetic were the analogues of lines and points in geometry, Frege would have ended up with a sort of Kantian conception of arithmetic, believing that the arithmetical axioms are *sui generis* and irreducible (like the axioms of geometry). But that, of course, was not his view. The task of defining the natural numbers (and the concept of a natural number) was designed to prevent this result. It was designed to show that the axioms are not *sui generis* in this manner. Hence, the numbers are the analogues of directions in geometry rather than of lines. As directions can be defined in terms of "basic" geometrical objects – lines, objects whose mode of being given justifies the axioms – so numbers can be defined in terms of "basic" logical objects – courses of values of functions, objects whose mode of being given to us justifies the basic truths of the domain.

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[24] Cf. Boolos (1990) p. 248. Boolos argues that the Fregean analogy between "the direction of l" and "the number of Fs" is misleading, because "we do not suspect that lines are made up of directions, that directions are some of the ingredients of lines". This is another way of saying what I argue in the text – that directions are not basic geometrical objects. But in contrast to Boolos, I believe that this substantiates Frege's analogy, as the point of these sections in FA is to show that numbers are not "basic" objects either – they are definable in terms of the logical objects. Hence the analogy does serve a point.

(E) Digging into the Self-Evident

In order to sharpen the edges here, I shall compare some aspects of the above account with some recent alternatives. Fundamental logical principles (rules of inference as well as axioms) were considered self-evident by Frege. And this has been presented by many authors as their ultimate justification. There is no going beyond this point, according to this conception; this is the end point of the justification-game. Some people regard any attempt to step "beyond" this point as betraying a misunderstanding of Frege's conception of the "autonomy" of logic, the view that there is no "meta-logical perspective" (to use Ricketts' suggestive expression; see his 1986) from which the fundamental logical truths may be justified. It may seem that our previous account about the role of sense in justifying the basic truths of a domain (including logic) is opposed to that, as if we have suggested an epistemological perspective (grasping the ways in which logical objects are given to us) precisely as such meta-logical perspective.

This may indeed be the case, if logic is narrowly construed as being concerned solely with deductions and proofs (as Ricketts and many others seem to suppose). On this narrow conception any attempt to view logical axioms as within the justification space, as being susceptible to any form of justification would amount to attempting a meta-logical perspective. I have argued, however, that Frege's conception of logic was broader: logic, on this broad conception, inheres in, and actually constitutes, the entire justification-space; it is concerned with whatever is relevant to justification. If this broad use of "logic" is adopted (in line, I believe, with Frege, as well as many other nineteenth-century logicians), and logic is construed as the science of justification and objectivity, then the theses of the autonomy of logic and the lack of meta-logical perspective are correct. But then the sense-oriented justification of the basic logical principles and an account of their objectivity are within the logical enterprise. It is not a step beyond logic.

Tyler Burge has rightly emphasized Frege's epistemological concerns in this connection (see Burge 1992, pp. 645-649). However, he characterizes the self-evidence of the basic logical truths in terms of understanding: "[Frege] did see [the basic logical truths] as a source for the justification of the belief in them by a *person who understood them*" (645, my emphasis – G.B). Burge then (rightly again) connects "understanding logical truth" with "understanding the nature of justification for our mathematical judgments" (646) and tries to explain this in terms of Frege's late conception of logic in *Der Gedanke,* as the science of the laws of truth ("what is") and as the normative science of the laws of thought. The connection is couched in very general terms – the general notions of reason and understanding – and as such seems to me of limited explanatory value. In any case the line I suggest here is much more

specific in its reliance on a fully fledged notion of sense as a mode of presentation, and on the two faces of objectivity – justification, on the one hand, and relation to objects, on the other.

Burge expands on these themes in his more recent paper of 1998. Frege proposed ways of justifying the basic truths of his logic in what Burge calls "justification through application" (Burge 1998, pp. 330-335). He rightly comments that these applications serve as partial, inductive "tests" of the logical system (1998, p. 332). With regard to the axioms he says that "the recognition of advantages seems to provide a prima facie, probabilistic justification [...] Such recognition may provide indirect grounds for believing that the axioms are indeed basic and indeed true. But the supposed self-evidence of the axioms is ideally the primary source of their justification" (334). The self-evidence of the axioms is not a subjective, or even an epistemic conviction. It is an objective property concerning which we are fallible, as we may wrongly judge a proposition to be self-evident. We may be wrong either in ascribing self-evidence to a proposition that lacks it, or, inversely, in failing to recognize it where it inheres. And if a proposition is self-evident we come to recognize this by a thorough understanding of it (339).

Understanding a truth may come in various degrees and may involve various factors, including, e.g., the inferential relations it has within a system. But what I am suggesting here is that a thorough understanding of a basic truth must also include a feature of its justification (or grounding), which consists in its expressing the ways in which its objects present themselves to us – their sense. Frege says that "the truth of a logical law is immediately evident from itself, from the sense of its expression" (CP 405). We should take "sense" here very seriously. Understanding a law is grasping its sense, the thought expressed by it; and a thorough understanding is a thorough grasping of the sense, which includes knowing the way it is built up by its constituent senses. These, in typical cases, include the modes of presentation of the objects of these thoughts. Thus recognizing the truth of an axiom and justifying a basic truth are grounded in grasping the senses of their constituents, the modes of presentation of their objects.

Burge points out that although, in some sense, Frege presents arguments for his axioms (e.g. in section 18 of BL), these arguments are neither semantic, nor justificatory. They are not semantic in that they don't mention symbols, and they are not couched in a semantic meta-language. They are not justificatory in that they do not provide a justification of the axiom on the basis of other, more basic axioms. How then should these explanations or arguments to be regarded?

They should be regarded, Burge claims, as explications or articulations of the meaning of the axioms, as expressing a proper understanding of them (1998, pp. 316-

329). I basically agree with all that, but wish to emphasize a point that seems to me crucial. With regard to his first axiom, a⊃(b⊃à), Frege argues, or explains, that "it could be false only if both a and b were the True while a were not the True. This is impossible. Therefore a⊃(b⊃à)" (BL, section 18). Comparing this to its "counterpart" passage in *Begriffsschrift*, Burge concludes that "The argument serves to articulate understanding of the thought content. It does so in a way that enables one to recognize that its truth is guaranteed by its content" (1998, p.317). This may be an apt description of the *Begriffsschrift* passage, but the BL one contains more. On the face of it, there does seem to be an argument there: Frege articulates what would be required for the axiom to be false, in a way that makes it manifest that this is impossible, and concludes that it must be true. Burge may be right that this is not a semantic argument, and that it does not prove the axiom on the basis of other truths. What then is the point of the argument? The argument, I suggest, couched in the object language, highlights features of the ways the objects concerned (the two truth-values) are given to us (ways constrained by the principles of the classical truth tables presumed here) as justifying the axiom, and laying its truth open to view. Frege does not merely appeal to the meaning of the axiom and to our understanding of it; he appeals to it in a particular way that makes manifest the way its objects are presented to us by appealing to the principles constraining this way.

The justification relation between senses and basic truths is not a foundational one. It is not that senses are prior to or more basic than logical principles. The principles express aspects of the senses of the fundamental objects of the domain, whereby they (or our beliefs in them) are justified.[25]

Justification is not construed here as what one might call a purely epistemic notion: It is not concerned only with the question of how a belief in the truth of a certain principle is justified, but also with the (ontological) question of what such a truth amounts to. On this Frege held a firmly realistic view according to which such truths are about objects and their properties and relations, and this, for him, was an essential feature of their truth. The justification I am talking about concerns modes of presentation of these objects and properties – their senses – in the fully fledged senses of these notions. I emphasize this in order to distinguish this position from others such as Peacocke's (1992), which seem to avoid such realistic commitments to objects and their modes of presentation. Peacocke presents a theory that connects what he regards

[25] P. Simons pointed out that Frege's note 16 to section 10 of BL, when read in light of FC, suggests that "the two sides of [Axiom] V express the same sense but in different ways" (Simons, 1992, p.765) The fact that Frege could think so fits the non-foundational character of his conception of justification alluded to in the text.

as a Fregean notion of sense with justification. But, leaving aside for the moment the connection itself, it seems to me that the notion of sense alluded to is not the fully fledged Fregean notion of mode of presentation. What Peacocke presents is in fact a version of a Davidsonian conception which in this respect is clearly non-Fregean.

Peacocke's idea is that a system of "understanding conditions" or "possession conditions" for a certain expression (conditions accepted by anyone who understands the expression) determine a "semantic value" for that expression. In fact Peacocke assumes that under certain requirements it determines the semantic value uniquely. The semantic value may then justify principles including the expression, in the sense that, given the semantic values of the expressions involved, the principle may be shown to be correct (803; section 4). Now, Peacocke realizes that such semantic values are not Fregean senses – they have different modes of presentation and may be given to us in various ways. Peacocke assumes, however, that the main effect of Fregean sense can be accomplished by a theory of "canonical derivation", according to which, e.g., the truth conditions of a sentence may be "canonically derived" from the semantic values of its constituent expressions. But such a theory, if it can be worked out, would not actually be using Fregean senses. On the contrary, it is designed to bypass them, to show them to be dispensable. For Frege, in contrast, Fregean senses – modes of presentation of objects and functions – are indispensable in an account of the objectivity and justification of the basic truths of any domain, including logic.

Chapter 4: Identity in Frege's Begriffsschrift

(Where Both Thau-Caplan and Heck Are Wrong)

Frege's views on identity continue to provoke scholars, and rightly so. In particular his view in *Begriffsschrift* (BS) of 1879, and its relation to his view in "Über Sinn und Bedeutung" (SR) of 1892 deserve careful attention. The issues involved have a wider significance than Frege's specific views on identity in different periods, though these are important enough. They concern also the move from what I call below "thin" semantics, which is exhausted in signs being assigned content, to a "thick" semantics, in which "modes of determining their content" (BS), or *Sinne* (SR) are also concerned.

In "What's Puzzling Gottlob Frege?" (2001) Michael Thau and Ben Caplan (T&C) argue that in SR Frege did not reject his earlier view of identity in BS, and that the arguments raised there endorse those of BS. In this, I believe they are right[1]. However, T&C adopt and enhance a widespread view, according to which the BS theory of identity is "meta-linguistic". By this they presumably mean that identity is a relation between signs and that an identity statement is about signs. This, I believe is wrong, or at least misleading. I shall argue that the objectual vs. meta-linguistic opposition is anachronistic and may be misleading with regard to BS, in a way that can distort its view. If, nevertheless, we insist on enforcing it on the text, the BS view should rather be seen as objectual, not meta-linguistic in the above sense: Identity, in BS, is **identity of content**; and the contents of names are objects. But in order thus to talk of objects, Frege realized, we must regard identity statements as concerned **also** with the ways of determining these contents. So, at least, I shall argue.

T&C also argue that the meta-linguistic theory they find in BS is never rejected or replaced by Frege in his later writings. I think this is wrong, but shall not discuss the matter here. I confine my remarks here only to some aspects of BS and SR.

In his "Frege on Identity and Identity Statements" (2003), Richard Heck accepts T&C's meta-linguistic construal of identity in BS, on which, as stated above, I believe he is wrong, but rejects their view with regard to Frege's later writings. Heck insists on distinguishing the question of what identity is, from the question of what identity statements express. In BS, according to him, Frege thought that identity is a relation

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[1] I have argued so myself in the section on identity in the Introduction to my Hebrew translation of BS (Jerusalem: Shalem, 2003). I didn't know then of T&C's paper, and had argued for this view in lectures many years before that.

between expressions; later (from the early 90s), he rejected it and thought that identity is a relation between objects. Heck seems to begin the late phase (as is customary) with "Funktion und Begriff" (FC), from which he cites extensively. This should include SR, which was published after FC, but Heck doesn't discuss SR in detail.[2] As to what identity statements express, Heck is less clear. He claims that in *Basic Laws of Arithmetic* of 1893, (BL), Frege thought that '0=1' for instance "denote the True if and only if zero is one. The thought it expresses is thus that zero is one, not that '0' denotes the same object as '1' "(95). This is of course right, but it may seem to underrate the main point, viz. that the thought expressed here is not just the identity of the objects, but their identity **as conceived under the particular corresponding senses** (of 'zero' and 'one'), which are constituents of the thought.

The Main Point of the Identity Section (8) in BS and Its Relation to SR

Both T&C and Heck (as almost everybody) seem to ignore a terminological point that reflects Frege's main innovation in the identity section (8) of BS. Frege distinguishes there between names (*Namen*) and signs (*Zeichen*). The distinction is not explicitly stated, but it is used almost consistently in section 8. In the second sentence of the section, for instance, Frege writes:

> "Während sonst die **Zeichen** lediglich Vertreter ihres Inhaltes sind […], sobald sie durch das Zeichen der Inhaltsgleichheit verbunden werden; denn es wird dadurch der Umstand bezeichnet, dass zwei **Namen** derselben Inhalt haben".
>
> ("Whereas in other contexts **signs** are merely representatives of their content […], they suddenly display their own selves when they are combined by means of the sign for identity of content; for it expresses the circumstance that two **names** have the same content." (Emphases added))

Throughout the section, except for the last sentence, Frege speaks of identity consistently in terms of names, i.e. signs endowed with modes of determining their

[2] SR is quite surprisingly silent on the matter and does not explicitly answer its opening question. A sort of an objectual construal of identity is however naturally suggested, and wouldn't be off mark.

contents[3], whereas in the rest of the book he talks, where identities are not concerned, simply of signs. A sign, in BS, just denotes its content; this exhausts its meaning. A name, in contrast, includes a mode of determination (*Bestimmungsweise*) of its content. This notion of mode of determination of content is akin to the later notion of sense (*Sinn*), and forms the main innovation of section 8 of BS. It would be going too far, however, to identify it with the later notion of sense and to claim that Frege had the notion of sense already in BS. For first, in BS Frege appeals to mode of determination as an exception to his general semantics, an exception that is confined to identity statements, whereas his later notion of sense applies generally to any expression in any context. Second, the later notion of sense cannot be detached from the comprehensive and rich theory of sense and reference, which Frege had not yet have in BS. In any case, it seems evident, as we shall see, that by bringing in names and modes of determining their contents into his account of identity in BS Frege did not mean to deny that names (in identity statements) also denote their contents, and that such statements are about these contents. He wanted to say that **in addition**, they also include (and express) the ways in which the contents are determined.

The semantics of signs in BS is a "thin", one-dimensional semantics – signs stand for their contents, and this exhausts their meaning. The correlation between a sign and its content is an arbitrary convention or stipulation. Signs are not inherently or internally connected to their contents. Talking about a sign is, therefore, meta-linguistic in a simple and straightforward way. The semantics of names, in contrast, is "thick": a name does not only denote its content, but includes and expresses a way its content is determined. This, Frege emphasizes, is an objective feature that pertains to the "essence of things" (*Wesen der Sache*), to use his terms. The connection here is therefore not arbitrary or conventional; it is essential and objective. This distinction is the heart of one of Frege's main arguments there, namely, that identity statements are **not about signs** conceived as forms: "Die verschiedenen Namen für denselbe Inhalt nicht immer blos eine gleichgiltige Formsache sind" ("Different names for the same content are not always merely an irrelevant question of form"; BS 15). A name, including as it does a mode of determining its content, is essentially connected to its

[3] Heck's ignoring the distinction between names and signs reflects in a mistranslation in which he says that "Names 'stand at times for their content, at times for themselves' " (Heck, p. 89). Frege writes Zeichen where Heck puts "names").

content. Talk of names is therefore also talk of their contents and talk of names (in identity statements) is not simply meta-linguistic. The distinction between the linguistic and the meta-linguistic may thus be misleading here (it is definitely anachronistic with regards to BS).

Having said that, in order to gear the following remarks to the articles of T&C and Heck, I shall distinguish in what follows between "the simple meta-linguistic view" – that identity statements are only about linguistic **signs** – and "the complex meta-linguistic view" – that identity statements are about **names** and their contents, where a name includes the mode of determination of its content. In using a sign in a (regular) proposition, according to BS, the sign just goes proxy for its content (say, object), and the connection between them is arbitrary and conventional. In using a sign, we talk about its content, but since a sign is not essentially related to its content, and the relationship between them is conventional and arbitrary, talking of a sign does not pertain to its content, and is thus simply meta-linguistic. In using a name in a proposition of identity, on the other hand, it stands for the content **and expresses the way it is determined** (Frege says it "includes" it); the connection between name and its content is therefore not arbitrary, but relates to the "essence of the thing". Talk of the name is thus complexly meta-linguistic. The main point of section 8 of BS is the move from the simple to the complex view. Frege is up to show that the simple view is "an empty illusion" (he speaks of the Nichtigkeit dieses Scheines). Frege introduces his explanation for this by saying:

> "Dies erweckt zunächst den Auschein, als ob es sich hier um etwas handle, was dem *Ausdrücke* allein, nicht *dem Denken* angehöre.[...] Um die Nichtigkeit dieses Scheines klar zu legen wähle ich folgendes Beispiel aus der Geometrie (14).
>
> ("At first we have the impression that what we are dealing with pertains merely to the expression and not to the thought […] To show that this is an empty illusion, I take the following example from geometry").

This is a clear rejection of the simple meta-linguistic view. Frege then gives a geometrical example of an identity, and explains that in understanding it we must consider not only the signs and their contents but also the ways these contents are determined. A major point here is that this view does not preclude that the relation of identity is between contents (objects) and that identity statements are about contents.

This characterizes what we have called the complex view. Frege's view there cannot be understood without the idea (perhaps obscure and, some would say, confused) that in talking about modes of determination of a content (included in names) we may *ipso facto* also talk about the content; and a relation between modes of determination of contents may thus be also a relation between the contents.[4]

This conception doesn't seem to fit squarely within the current linguistic/meta-linguistic distinction. But this should not detain us, for there are a lot of anachronisms that get into the way both T&C and Heck present the simple meta-linguistic view. This perhaps is inevitable, but it is important to be aware of it, and take the required precautions. The very idea of a meta-language is foreign to Frege of 1879. Heck also talks about identity in BS as a function, and asks whether the arguments of this function are names or objects. But this has an anachronistic ring about it as well. For Frege of BS, functions and relations, as well as arguments, are linguistic expressions, not the contents of them:

> "If in an expression, whose content need not be capable of becoming a judgment, a simple or a compound sign has one or more occurrences and if we regard that sign as replaceable in all or some of these occurrences by something else (but everywhere by the same thing), then we call the part that remains invariant in the expression a function, and the replaceable part the argument of the function" (BS, section 9).
>
> ("…so nennen wir den hierbei unveränderlich erscheinenden Theil des Ausdruckes Function, den ersetzbaren ihr Argument").

It seems apparent (though the point has been debated) that here both function and argument are linguistic expressions.[5] Hence, for Frege of BS it was most

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4 This, I believe, is true also of his later notions of sense and reference. I argued for that in Bar-Elli (1996), both as a general thesis, and in particular with respect to his view of oblique contexts and *ungerade Bedeutung* (see ch. 9).

5 It is doubtful whether Frege has been entirely consistent in this. Some scholars assume that function and argument are linguistic expressions in BS (see for instance Carl (1996), p. 65), while others claim that function and argument are contents in BS. Baker and Hacker, for instance, assume, without much ado, that function and argument in BS are contents (1984, 133-85); In their recent (2003) they criticize the "linguistic interpretation", which they present as the standard one (held, among

natural to speak of identity as a relation between names (including the mode of determination of their contents).

Frege in BS did not yet have the later distinction between sense and reference, and he talked in terms of the one notion of content (*Inhalt*). Content was presented as a sort of abstraction from judgment, where one abstracts from the claim to truth in a judgment (BS, section 2). Yet, Frege distinguished there between contents that can become a judgment (*beurtheilbare*), like the content of a propositional sign, and contents that cannot, like the content of a singular name or a functional expression. Since Frege later changed his view, and replaced talk of content by talk of either reference (*Bedeutung*) or sense (*Sinn*), it is difficult, and perhaps impossible to be strict here, but in most contexts in BS his use of "content" is much closer to his later reference than to his later sense. It seems clear, for instance, that in BS contents determine the truth and falsity of judgments, as well as the logical potential of inferential relations, which are entirely "extensional" in BS. It is also clear that in the geometrical example Frege gives at § 8 the point itself is the content of the name, and it is distinguished from the way of determining it. Hence it seems clear that at least in identity statements, the contents concerned are akin to Frege's later references – they are the objects themselves. Be it as it may, my point here against the simple meta-linguistic view, and for the claim that identity statements are also about the contents of their names, holds good even if we construe the notion of content somewhat differently.

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others, by Dummett, Geach and Kenny), and propose an alternative reading, based, *inter alia*, on a sharp distinction between "expressions in BS" and "expressions in a natural language" (284-8). But even they present their reading quite cautiously as a possible one, whose difficulties are outweighed by its alleged merits. I believe that Frege, though he may have been somewhat careless about use and mention in BS, was driven to his linguistic conception of function for weighty reasons, but I cannot expand on them here (see note 12 below). Like Demopoulos I believe that "function" of the first chapter of BS should be distinguished from "procedure" (*Verfahren*) of the third, though Frege use the same letters for both (cf. Demopoulos (1995), note 26 on p. 85). But these niceties are impertinent to the question discussed in the text, and I shall not get into them here.

The identity sign can occur between expressions for both kinds of content – those that can become judgments and those that cannot. This raises a problem: In his later view Frege regarded a sentence as a complex name referring to a truth value, regarded as an object.[6] It was then straightforward to regard propositional equivalence as identity. In BS, however, Frege did not have the notion of a truth-value as an object denoted by a sentence, as in his later view. And yet he had one notion of identity applicable to all contents (or expressions). In particular, he did not distinguish between identity and propositional equivalence (we shall ignore here the problems pertaining to identity of functions). Since his main point in section 8 is that in identity statements we must consider not only the content (and not only the sign for the content), but also the way the content is determined, this must apply to the propositional case as well. (Had Frege been consistent here, his view that the identity sign may occur between sentences, and that in general the identity sign occurs between names, should have led him to regard sentences as names already in BS.) But whereas we may have a fairly clear idea of the distinction between an object and the way it is determined, we seem to lose any grip on the idea with respect to propositional contents: what could the distinction amount to there?

We can grasp the distinction between the man Socrates and the way of determining it as the content of an expression, which is included in the name "the great teacher of Plato". Can we grasp, analogously, a distinction between the content of, say, "the earth is round" and a way of determining this content, which is "included" in the sentence? We can grasp of course the distinction between a content and a way of determining its truth. But this is not the issue here. The issue is whether we can intelligibly distinguish between a propositional content and a way of determining it (not its truth). The difficulty in providing any satisfactory account for this was, I believe, one of the main reasons for Frege's later view, in which he abandoned the notion of content in favor of the bifurcation into sense and reference, and postulated truth-values as objects named by sentences, and thoughts as their senses. But I shall not elaborate on the point here.

For these and other reasons, it may seem pointless to be very strict about Frege's formulations in comparing his view of these matters in BS with his later views. Heck

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6 I expanded on the reasons for recognizing truth-values as objects in Bar-Elli (2001), here chapter 3.

and the "conservative" interpreters, may justly claim that so much the better for their view that much of this changed in the 90s. But for understanding BS we should better try seeing through this haze of inaccuracies and anachronistic terminology into the real issue. And the real issue, I suggest, is the recognition, in BS section 8, that the "thin" semantics of signs standing for their contents is too narrow a framework for the semantics of identity statements; that we must also consider modes of determination of contents, which are "included", or expressed by the names for them. The semantics of names (in identity statements) is thick in that they do not only have contents and stand for them, but also include (and express) modes of determining these contents. These modes are objective, they are communicable and belong to the "essence of things", and they are expressed by names, in addition to these names denoting their contents. Let me emphasize again that in BS Frege had not yet generalized this view to all expressions in all contexts – this, with a comprehensive theory of sense and reference was to come later, beginning in the articles of the 1890s. In BS, his general semantics, except for identity statements, remained "thin".

The Identity Relation, and What Identity Statements Are About

In the opening of section 8 of BS Frege says that a statement of identity is about names, not about their contents. This is an unhappy formulation. Frege evidently meant that it is about names, and not **merely** about their contents. This should be evident not only from the title of the section (*Die Inhaltsgleichheit* – The Identity of Content); it is repeated in the body of the section, where Frege speaks repeatedly of the sign of identity as a "sign for the identity of content". This would be most implausible if Frege thought that identity were not a relation between contents. The content, as mentioned above, can be one that can become a judgement (*Beurtheilbare*) – by his later view, a thought – or one that cannot – in his later view, an object or function.

That identity statements are also about the contents of their names is evident also from the main tenor of the section, where Frege argues that a mode of determination of a content is not a matter of arbitrary stipulation concerning signs and their form, but has to do with the "essence of things":

> "Die verschiedenen Namen für denselben Inhalt nicht immer blos eine gleichgiltige Formsache sind, sondern dass sie das **Wesen der Sache**

> selbst betreffen, wenn sie mit verschiedenen Bestimmungsweisen zusammenhängen" (p. 15)
>
> ("Different names for the same content are not always merely an irrelevant question of form; rather, they concern the essence of the thing, if they are associated with different ways of determining the content". (I changed here the Bauer-Mengelberg translation).

This means that there is nothing arbitrary or conventional in this connection, and since a name "includes" (or expresses) such a mode of determining a content, its connection to the content is also not arbitrary or conventional. This, to repeat, is not a subsidiary casual remark, but the gist of Frege's argument in the section. Earlier in the section he says that the possible impression that identity is a relation between mere arbitrary signs is "an empty illusion". Hence, the simple meta-linguistic theory is not Frege's view in BS.

Moreover, it can't be his view unless one is prepared to ascribe to him (in the work in which he initiated modern logic!) the most elementary fallacy of ambiguity. For if we take the simple meta-linguistic theory literarily, from "Fa" and "a=b" one cannot infer "Fb", since the two occurrences of "a" (and the two occurrences of "b") do not have the same content (do not refer to the same thing) – the first refers to the content of "a", while the second – to "a" itself.

Frege, of course, endorses the law of identity – it is a fundamental principle of his logic (52), and he ends section 8 itself with it. It is hardly credible that Frege fell victim to this fallacy, and didn't notice that on his view of identity the principle turns out invalid, and that this view makes every "mixed" context, in which identity occurs with regular contexts, incomprehensible. Hence, for anyone who would rather not ascribe to Frege these trivial fallacies, the simple meta-linguistic theory cannot be Frege's view in BS.[7]

And in fact, as stated above, a close reading of section 8 of BS makes it clear that Frege's main concern is to show that identity statements are not only about the contents of their signs (as is the case in e.g. conditional and negative statements), and not only about the signs (as the simple meta-linguistic view holds), but about the contents conceived by the modes of determination included in their names. One needs

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[7] This is different from, though related to, Heck's claim that the meta-linguistic view of identity makes quantified identity statements muddled (87).

not adhere too stringently to the unhappy formulation of the first sentence (cited above), but try to understand the main tenor of section 8 of BS.

We thus see that there are strong reasons to believe that in BS Frege thought that identity statements are concerned with the contents of their names. On the other hand we have Frege's seemingly conflicting claim that identity is a relation between names. The way to reconcile these two, I have suggested, is to realize that a name, being endowed with a way of determining its content, is essentially and non-arbitrarily connected to its content, for the way of determining it is objective and pertains to "the essence of the thing". Hence, we can say, as I said above, that when names (not merely signs) are concerned, talk of names may also be talk of their contents.

Heck (like T&C and pretty much everybody)[8] claims that the BS' theory of identity is simply meta-linguistic, and emphasizes, in support, Frege's wording of his (52), in which Frege writes: "the content of c is identical with the content of d" etc. (Heck, 95). But this is poor evidence. First, "the content of c" is a complex name of a content, not of a sign. Second, Frege writes in this way throughout BS, with regard to all contexts, including regular ones, like simple monadic formulas, or conditionals, negations, etc. But these contexts are explicitly about the contents of their signs, and no meta-linguistic theory with regard to them is at issue at all. Even in the above wording of (52), which Heck quotes, Frege renders a simple conditional like "If f(c) then f(d)" in his regular way in terms of "the case where f(c) is affirmed and f(d) is denied" etc. A careful reading of the explanation attached there suggests that "f(c)" and "f(d)" are mentioned here. On Heck's reading, if followed consistently, this would imply that simple monadic predications and conditionals should be read "meta-linguistically". But no one would suspect Frege of proposing a meta-linguistic construal of regular contexts like "f(d)", or "if f(d) then g(c)" etc., and he explicitly

[8] Let me mention one more example. W. Taschek (1992) presented Frege's "identity puzzle" in a way with which I am, in general, in sympathy. But with regard to BS he also subscribes to the mistaken simple meta-linguistic view (see pp. 770-771). This, I believe, hardly squares with his (justified) opposition to information-based rendering of content, and with his view that this rendering implies a meta-linguistic solution to the puzzle.

says at the beginning of section 8, that in these contexts signs simply stand for their contents, so that in using them we simply talk about their contents.[9]

We can use the following list of distinctions for summarizing some of the above points:

1. The identity relation is a relation between:
   a. Objects.
   b. Signs.
   c. Names.
   d. Contents (objects) conceived under a particular mode of determination contained in their names.
2. An identity statement expresses the thought that:
   a. Two signs refer to the same thing (have the same content).
   b. Two names (including the mode of determination) refer to the same thing.
3. An identity statement expresses the thought that:
   a. The content (referent) of a sign is the same as the content (referent) of another.
   b. The content (referent) of a name (including the mode of determination) is the same as the content (referent) of another.
4. An identity statement is about:
   a. The names it contains.
   b. The contents (objects) referred to by these names.
   c. The contents (objects) as conceived under the modes of determination included in their names.

For the reasons expounded above, I propose that 1.d., 2.b. (or 3.b. I take them to be equivalent)[10], and 4.c. are the real views expounded in BS. We have found four main

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[9] In general, it is often difficult to decide whether affirmation and denial are meant to apply to contents or to propositions (Sätze). A careful reading of §5, for instance, suggests that they apply to propositions. This may be one of the cases in which reading BS with a strict distinction between use and mention may be quite embarrassing.

[10] They are not strictly equivalent, for in 2 names are directly mentioned, while in 3 they are indirectly mentioned in mentioning their content. I still take them here as equivalent, in line with the view, expounded in the text, that since a name includes a

reasons for that: (a) Frege repeatedly talks in section 8 of identity of content, which is also the title of the section, and which would be most implausible if he thought that identity was merely a relation between expressions. (b) A Name, in distinction from a mere sign, includes a mode of determining its content, so that the connection between them is not arbitrary, which may render talk of names to be *ipso facto* talk of their contents. (c) The simple meta-linguistic view does not square with Frege's use of the law of identity, which is basic in his logic, and with "mixed contexts". (d) Frege's basic argument in section 8 would not make much sense on the simple meta-linguistic model, since he explicitly claims that signs (unlike names) are connected to contents arbitrarily, and his explicit and main point was to overcome this arbitrariness.

I believe that each of these makes a good case for my point here; all of them together make it overwhelming: In BS Frege did not hold the simple meta-linguistic theory; if we are to enforce the meta-linguistic perspective on his views, we might say that he held the complex one: identity, being a relation between names, is also a relation between their contents, conceived or determined in particular ways, and an identity statement is about these contents.

The Change in SR

True – this, strangely enough, was confined in BS to identity statements. But since identity is a logical notion, and is part of the basic notation of BS, the recognition must have had a general significance, the full realization of which came in the early nineties. With removing this confinement to identity statements, the gist of the BS view of identity is maintained in SR. It is in fact generalized there as pertaining to the entire language – to expressions in any context, not only to identity statements, as in BS. SR presents many other important aspects – indeed a whole theory – of sense and reference, which are beyond our concerns here. I confine my remarks here only to the particular distinction between content and mode of determining it, which was shown to be crucial for understanding Frege's theory of identity in BS. In SR the distinction between signs and names is rejected – all meaningful expressions are treated as including or expressing modes of determination of their contents (i.e. on the model of "names" in identity statements in BS), and thus a uniformity of the semantics in all

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mode of determining its content, in speaking of a name, one indirectly speaks of its content.

contexts is attained, though the semantics is uniformly a "thick" one. SR could thus be regarded as a generalization of the BS semantics of identity to all contexts.[11] What is rejected in SR of the BS account is the "thin" conception of the meaning of expressions (signs) in regular contexts (in which signs just stand for their contents), not the "thick" conception of names in identity ones (in which they also include the mode of determining their contents).

The result of all this was the rejection, in SR, of the BS notion of content (*Inhalt*), in favor of the bifurcation into reference (or meaning, *Bedeutung*) and sense (*Sinn*). A major reason for this was Frege's realization that the general "thin" semantics of signs and contents in regular contexts in BS could not allow for the vital distinction, recognized already in BS with regard to identity statements, between content and a way of determining it: If one says that the latter is part of the former (included in it), identity statements carrying cognitive value would not be true, for their contents would be different; if, on the other hand, one says that it is not part of the content, it has, within the confinements of "thin" semantics, no place and no role to play in understanding these identity statements. Frege's argument shows that both horns of the dilemma are unacceptable and we have to abandon the restricted limits of the thin semantics, and thicken it by the notion of modes of determination or Sinn. This, together with the above difficulty of applying the distinction to sentences and their contents were, I propose, among the chief reasons that led to the change.[12]

The "Identity Puzzle", Cognitive Value and Conceptual Content

I emphasized above that with regard to identity statements, Frege proposed in BS a "thick" conception in which names (not mere signs) include the way of determining their content. Now, in order to enhance the above dilemma, let us look in more detail into the possibility that this "mode of determination of a content" is part of the

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11 This perhaps explains Frege's strange formulation, in SR, where he repeatedly talks of "names or signs", as if to blur his own previous distinction between them.

12 Another major reason, which I shall not discuss here, was, I propose, somewhat similar difficulties Frege faces with regard to the notion of function in BS. Here again it was necessary to make some distinctions, which the narrow scope of the thin semantics of signs and contents in BS did not allow. I expanded on this in the Introduction to my Hebrew translation of BS ibid.

content. We have seen above that it is not arbitrary or conventional, but pertains to the essence of things. Yet, it is clear that in BS Frege did not regard it as part of the content. Supposing that he did, would make almost everything he says about content, particularly in section 8, unintelligible: For example, in the geometrical example he gives there point A and point B are proved to be the same; the content of "A" and of "B" is the point itself, and they differ in the ways it is determined as their content. If these ways were part of the content, their contents themselves would be different.

One must remember, as Frege advises the reader, that not any aspect of the content, but only "conceptual content" is concerned in a *Begriffsschrift* (BS, Introduction, and section 3). Conceptual content is defined as what matters for inferential relations. Whatever is not pertinent to these relations is not part of the conceptual content, and hence, is not logic's concern. In BS Frege apparently did not appreciate that mode of determination is thus relevant to inferential relations (he doesn't mention any that would be thus affected), and therefore did not regard it as part of the content. This may explain his bizarre and ad hoc attitude, in which he marshaled the notion of mode of determination in order to account for what seemed to him then a peculiarity of identity statements.

Similar remarks apply, as we shall see, to cognitive value. Suppose we grant that two sentences, including different names, which denote the same thing but contain different ways of determining it, differ in cognitive value. Does this reflect in their inferential power? There isn't any mentioning of this in BS.

It seems therefore that only in SR did Frege discover that there are contexts in which the mode of determination (what he then called *Sinn*) must be regarded as belonging to the conceptual content – namely, oblique contexts of "indirect reference", like belief contexts. Here *Sinne* must be brought into a proper account of inferential relations.[13] There are uses of 'belief' in which from 'x believes that Fa' we

[13] I find it quite surprising that Taschek says that "Frege nowhere offers a sustained and systematic discussion" of the logical relevance of cognitive differences of the kind he discusses (772), for this is a major topic in SR (particularly 37-39.). The discussion there seems to me more directly pertinent to the logical appraisal of differences in cognitive potential than the indirect way by which Taschek argues in section 3 of his paper.

can infer 'x believes that Fb' only where 'a' and 'b' have the same *Sinn* or mode of determining their contents.

The introduction of *Sinne* is usually tied up with the "identity puzzle" Frege presented in both BS and SR. Heck considers three versions of the puzzle (88-90):

1) a=b is informative although a=a is not (they differ in cognitive value).
2) Fa and Fb differ in cognitive value, although ‘a’ and ‘b’ refer to the same thing.
3) ‘P’ and ‘Q’ may differ in cognitive value, although they denote the same truth value.

I shall not go here into the differences between these versions, and the question of which is Frege's. A long and venerable tradition has presented the "puzzle of identity" that brought Frege to his celebrated distinction between sense and reference in SR, in terms of "cognitive value". Heck rightly emphasizes that the issue is not the difference between a sentence that has and one that does not have cognitive value; the issue is rather that two sentences with co-referring terms may have different cognitive values (89-91).[14]

He also argues that according to Frege "the thoughts expressed by certain sentences differ **because** the senses of the names contained in them differ" (93). Evidently, there are contexts in which this "because" is plain and straightforward: a thought consists of its constituent senses; hence a change in these senses causes a change in the thought. However, as a general claim about difference of sense (which is evidently Heck’s intention here), this, I believe is a mistake. For Frege, the main criterion for difference of sense is difference of thoughts (containing the senses), and the criterion for difference of thoughts is epistemic: thoughts differ if a competent speaker can believe the one without believing (or when disbelieving) the other. Hence, if Heck's emphasized "because", in the above citation, is epistemic, if he maintains that we know that two thoughts differ because we know some of their constituents to differ in sense, he is, I think mistaken. It is, in general, rather the other way around: If two thoughts differ, and they are expressed by two sentences whose sole difference is that the one contains an expression where the other contains a different one (possibly co-referential), these two expressions differ in sense.

[14] This has been argued in Bar-Elli (1981).

This seems to me important, among other things, for the following reason: For Frege, sense belongs to logic, and logic, as stated above, is concerned only with "conceptual content" – namely only with those features of sentences that make a difference to their inferential power (what can be inferred from them, together with other propositions, and from what they can be inferred). Now, it could be argued (and has been argued) that the notion of sense and differences of cognitive value may be interesting and important for, say, epistemology and philosophy of mind, but do not belong to logic, and are not a basis for a logical difference.[15] This, of course, was not Frege's view. He, therefore, must have pointed out contexts in which sense and cognitive value matter for inferential relations. And Frege's paradigm examples for these are epistemic cases of propositional attitudes, like propositional belief contexts. This, which perhaps had been hinted at but not been fully appreciated in BS, was enunciated in SR. And with this appreciation came the full recognition of *Sinne* as belonging to the conceptual content and their full insertion into his logic. And note that sense, in these last considerations, is the "core idea" of a mode of presentation or of being given.

[15] Cf. the discussion of the "neo-Russellian" view in Taschek (1992), pp. 778ff.

Chapter 5: Logical Structure and Intentionality – Frege and Russell on Definite Descriptions

How are we able to think and talk successfully about objects in the world? This question has had a place at the heart of philosophical discourse from its beginnings.

At a rough approximation, philosophical views about how thought can be about things in the world can be divided into three main approaches I briefly mention without going into them. The first claims that a thought is about something if it includes that thing as a constituent. We succeed in talking about something in expressing (and grasping) a thought about it. The second approach holds that a thought is about something by virtue of having a constituent that "represents" this thing. The third approach claims that the question itself is based on an erroneous conception of thought and speech and the relation between them and the "world".

In his classic paper "On Denoting" of 1905 (henceforth, OD), Russell proposed a logical theory which was intended to explain how we use general concepts (such as "king", "mass", "center") and descriptions formulated with general terms, in order to think and talk about objects, and how our thoughts "reach" the objects that the propositions are about. Thus we can think about Hussein through the description "the present King of Jordan"[1] and about a particular point in space through a description such as "the center of mass of the solar system." Russell's theory is an important version of the first approach listed above. Like many others, Russell assumed that it is relatively easy to understand the possibility of thinking about an object when this object is a constituent of a proposition which we grasp. A great many of the thoughts we grasp, however, are not of this sort: The thoughts or propositions do not contain the object as a direct constituent, and the sentences expressing them do not contain terms that refer to it directly. According to Russell, his proposed theory explains the possibility of thinking and talking about objects even in these cases, as well as solving many problems which, in his opinion, alternative theories are unable to cope with.

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[1] The article was originally written when King Hussein was the King of Jordan.

I assume that Russell's theory ("the theory of definite descriptions") is generally known to the reader, and I shall not present it in detail here.[2] In brief, it analyzes sentences of the type "The F is G" as sentences of the type "There is an F such that there is no other F, and every F is G." Note that here the only constituents are the concepts F and G. "The F" does not appear and what it is supposed to denote is not a constituent. One of the alternatives Russell attacks in his paper is the one he attributes to Frege. Frege, however, as implied by his letters and posthumous writings, did not accept this criticism, nor did he consider it particularly valuable. Thus he did not accept Russell's theory either.

Why? Why did Frege reject Russell's theory of definite descriptions? We cannot be certain, since Frege never wrote anything explicitly about it, at least in the writings that have come down to us. Nevertheless, this paper presents what I think would have been his answer to this question. This involves fundamental issues in the philosophy of logic and language, especially about the connection between logical structure of propositions and what might be called their "intentional structure" (what they are about). The gist of my answer is my suggestion that Russell's theory, as it is generally understood and perhaps also as Frege understood it, does not explain how definite descriptions are conceived as singular terms, and that it does not provide a coherent explanation of the fact that descriptive sentences (sentences containing definite descriptions) purport to be directed at objects and be about them.

Nevertheless, I shall argue in the last part of the paper that Russell's theory should be understood differently, in a way that blunts the sting of the Fregean criticism and challenges some of Frege's basic principles about the connection between the intentional structure of a proposition (what it is about), and its logical structure. Thus the issue between Frege and Russell on definite descriptions involves the fundamental general questions mentioned above.

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[2] For a survey see Linsky (1977); Neale (1990), Chap. 2.

Descriptions and Terms Lacking Reference

Definite descriptions, such as "the present King of Jordan," are considered proper names (*Eigennamen*) in Frege's theory. As such, according to what is often taken to be Frege's "official" theory, they could have sense yet lack reference (see Frege SR, 28/58).

This aspect of Frege's theory, as he himself believed, is extremely problematic, since he claimed that the sense of an expression is the way its reference is given, and it is hard to understand how its reference can be given in any way if it does not have any reference. He also maintained that a thought is essentially true or false, which it could not be if it contained a referenceless constituent. Yet he considered his position to be the best possible approximation that could be given for the meaning of names in a natural language, many of which seem to lack reference. To be sure, Frege considered the fact that natural languages contain expressions of this sort to be an essential flaw in them (SR 40-1/69-70). A proper logical language should guarantee reference to all sense-bearing expressions (ibid.; TPF, p. 104; and many other places). This is indeed how Frege constructed his logical language in BL (cf. Dummett, 1981a, pp. 166-67). This is true also of the "description function" he introduces in BL §11,which forms a proper name when attached to a predicate, somewhat similarly to the definite description of natural language. The possibility of referenceless senses has been a hot issue of debate among Frege scholars and I shall not expand on it here.

Frege's strict position guaranteeing a reference to all sense-bearing expressions in a natural language may perhaps be defensible for proper names (in the ordinary sense of this term) such as "Hussein" or "Joe," since it is difficult to see what their sense could be if they have no reference. Even those who accept this defense, however (and there are very few such people) would find it very difficult to defend a similar position for definite descriptions such as "the present King of Jordan" or "the chess champion of the school." We would understand these expressions even if they had no reference – if there were no king in Jordan at present, or if no chess championship had taken place in the school, or if it had taken place but three participants had tied for first place, and so on. We must not

forget that definite descriptions form part of the structure of our linguistic competence: All speakers of a language can create and understand innumerable expressions of this sort, seemingly without any connection to their knowledge of the reference of the expressions.

It thus appears that Frege should have seen the problem created by his position that definite descriptions are proper names, and that an appropriate logical language should guarantee them a reference; he should therefore have been very happy to adopt Russell's theory of definite descriptions, which seemingly exempted him from this problem. However, he did not do so, and in spite of the fact that he never mentioned it explicitly, it seems probable that he actually rejected it. Our question of why he did so thus remains.

But could it be that Frege actually was unaware of Russell's theory? Although the theoretical possibility cannot be ruled out, it seems extremely unlikely. Russell was the first philosopher (at least outside of Germany) who acknowledged Frege's greatness and studied his work. He even devoted a long, detailed appendix to Frege in his *The Principles of Mathematics* of 1903 (henceforth PoM). Early in the twentieth century Frege and Russell conducted an intensive correspondence and exchanged papers (although always in German), and the topics they discussed included issues involving the meaning of proper names. It is thus hard to believe that Frege never read Russell's paper "On Denoting," which was published in *Mind* in 1905, since Russell considered the paper of revolutionary significance. Moreover, Frege's position is discussed and criticized explicitly and at length in this paper, which is the only paper published at that time to boast this position.

But although some slight doubt can be cast on Frege's acquaintance with Russell's theory of descriptions right after its publication, it seems certain that he knew it at a later time, as can be seen from his answer to a letter sent him in January 1914 by the British mathematician and historian P. Jourdain. In this letter Jourdain asks Frege explicitly if the distinction between sense and reference should not become superfluous in light of

Russell's new theory (see Letter VIII/11 in PMC 17-18).[3] Frege does not refer explicitly to OD or the theory of descriptions in his reply, but it is clear that this is what Jourdain's question was about. Frege does refer explicitly to Russell and Whitehead's *Principia Mathematica*, in which the theory of descriptions is presented in a revised version (see Letter VIII/12-13 in PMC 78-84). Frege writes in his reply to Jourdain that his English is not good enough for noticing nuances such as the analogy between Russell's theory of types and his own theory of orders. Yet it is clear from Frege's words that he read the *Principia* in English, and a reading of the entire letter (in both versions) makes one suspect that Frege's admission of the limitations of his English was somewhat ironical, and that he actually considered Russell's view unclear and even confused. At any rate, it is clear that Frege read English at a level where he could easily understand the principles of the OD theory, and it is thus implausible that he was unacquainted with it. However, neither then nor at any later time did Frege change his mind and accept Russell's theory. Thus we remain with our original question: Why?

In order to sharpen the question and see what exactly Frege did not like about Russell's theory, it is worth noting how close their views were in some respects. In BL Frege introduces a special function (symbolized by the slash \), which when applied to an extension forms a proper name, and defines it in a way that secures it reference: when the extension concerned is a unique object, it is the value of the function; when not – the value is the extension itself (BL §11). He realizes there that it is very close to many uses of the definite description of natural language, and notices that for genuine such uses the existence and uniqueness conditions must be fulfilled (cf. also his note I in SR, in FR 164/42). He repeats it in a critical review of a book by Schoenfliess, apparently written in 1906 (i.e. after OD):

[3] Jourdain's question, linking the notion of sense to descriptions in the way it does, displays, I believe, misunderstanding of Frege's notion of sense, but I shall not go into the details here.

> With the help of the definite article or demonstrative, language forms proper names out of concept-words.[...] If forming a proper name in this way is to be legitimate, the concept whose designation is used in its formation must satisfy two conditions: 1. It may not be empty. 2. Only one object may fall under it. (PW 178; cf. also ibid. 192).

Thus we see that even though Frege clearly saw the uniqueness and existence conditions involved in a descriptive proposition – conditions which are at the heart of Russell's theory of descriptions – he remained loyal to his own view from the 1890s (in SR and BL) and refused to accept Russell's theory. However, whereas on Russell's theory when these conditions fail, the descriptive proposition is false, which is also the case in Frege's formal language of BL, with regard to descriptions in a natural language Frege thought that it is then "illegitimate" to create the definite description. This may be plausibly interpreted to mean that in such a case no thought is expressed. And this seems to presume that the "real" logical structure of such a thought is that it is singular one about an object. As we shall see, this is a central issue for understanding Frege's position against Russell's theory of descriptions.

Alternatively, according to a widespread interpretation of Frege's theory, when these conditions fail the description is an "empty" name – having a sense but lacking a reference. The descriptive sentence then also has no reference (is neither true nor false). Yet, according to this interpretation it has a sense.

I believe that this interpretation is wrong and that Frege actually thought that sentences of this sort in a natural language do not express real thoughts, but are rather "mock sentences" expressing mock thoughts (*Scheingedanke*). I cannot go into this issue here,[4] but I would like to note that this position is supported by part of the quotation cited

[4] Admittedly, in many places Frege writes as if he recognizes thoughts (and the sentences expressing them) that lack reference, i.e., truth value, when they have constituents with sense but no reference. See SR 62/36; PW 122/133; 191-94/208-10; 198/219; 225/243; PMC 80;

above, as the above interpretation seems a rather weak one for the expression "legitimate" in this context. It seems that Frege's intention is that forming a definite description when the conditions are not satisfied is an invalid move that does not create a genuine name in the language. This may look quite a pricy position, for most descriptions may be such.

All this only intensifies Frege's problems in the face of his position that definite descriptions are proper names. Russell's theory, in contrast, contains no such demand for the "legitimacy" of a name – both of Frege's conditions do have to be satisfied in order for a sentence containing a definite description to be true, but if they are not both satisfied then the sentence is simply false, rather than lacking a truth value. Frege could therefore adopt Russell's theory as a natural way out of the dilemma at issue, and our question of why he rejected this theory remains acute.

Dummett bases the bulk of his answer to this question on the link between Russell's theory of descriptions and some implausible views of his, such as the theory of "logical names," which are such that whatever they denote must necessarily exist, or the view that ordinary proper names in natural languages are abbreviations of definite descriptions (see his 1981a, pp. 161-65). Many others follow this track.

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165. It would thus seem as though the existence of thoughts and the possibility of grasping them do not require them and their constituents to have reference and truth value. In all these places, however, Frege insists that there is no knowledge or "scientific" use involved, but only artistic or poetic uses. This seems somewhat puzzling: It is as if we said that the issue of truth and reference does not belong to the essence of a thought and the possibility of its existing and being grasped, but only to the use we make of it. I believe that this was not Frege's position: Truth and reference are at the essence of thought and the possibility of its existence. Thus a thought lacking truth value is an incoherent idea, and what seems to be such is not a genuine thought, but only a mock thought. I expand on this point in Chapter 3 of Bar-Elli (1996), and in Bar-Elli (2015), here chapter 6.

Without entering into a detailed analysis of these issues, it seems to me that this is a weak explanation, for two reasons: 1) The link between Russell's theory of descriptions and his views about logical terms and proper names in natural languages is not obligatory, and Frege could have adopted the first without having to adopt the others as well. 2) Frege himself was not so far from a view similar to Russell's on logical terms. Actually, as we have seen, something quite similar constituted his position on all meaningful names – that they all have a reference. It is thus hard to imagine that it was precisely this aspect of Russell's philosophy that Frege disliked. In fact, Dummett himself admits that Frege's view involves difficulties almost as severe as Russell's (ibid. p. 166).

The answer to our question involves, as I said at the outset, some basic issues in the philosophy of logic and of language, first and foremost the very conception of meaning, on the one hand, and the conception of logical structure and its relation to intentional structure, on the other. I shall now discuss these issues. First, I shall say a few words about meaning and then I shall concentrate on the issue of logical and intentional structure.

Meaning

To Russell himself, the principal significance of his theory of definite descriptions was that such expressions, like other denoting phrases, are "incomplete symbols" – that is, they are expressions without any meaning of their own, but which contribute systematically to the meaning of the sentences in which they appear. "The present King of Jordan is bald" is a meaningful sentence, and the definite description "the present King of Jordan" makes an important systematic contribution to its meaning, despite the fact that the description in and of itself has no meaning. (The word "systematic" in the preceding sentence is meant to express the fact that the description makes the same or a similar contribution to every sentence in which it has a function, in a way that can be explicitly described.) The meaning of a sentence thus does not lie in the attribution of a property to the object denoted by the description "the present King of Jordan", for if that were the case then we would not be able to ascribe a meaning to the sentence "The

present King of Israel is bald" (and we would also become involved in other problems which this is not the place to go into). The meaning of the first sentence, according to Russell, is something like "There is at least one thing such that it is presently King of Jordan and such that there is no other thing which is presently King of Jordan and such that it is bald." This can be represented in standard logical notation as follows:

$(Ex)(Kx\&(y)(Ky \leftrightarrow y=x)\&Bx)$

Russell shows that such an analysis can be performed for each sentential context in which the description "the present King of Jordan" appears. The analysis actually expresses the systematic contribution of the expression to the meaning of such sentences. Russell nevertheless insisted that the expression itself is totally devoid of meaning, and if we look carefully at the logical "translation" of the sentence we can indeed see that it has no element parallel to the description at issue – that is, it contains no expression denoting King Hussein.

This is so because Russell held that the meaning of an expression is not its contribution to the meanings of sentences including it, but is something in the world – generally an object – which the expression signifies.

In Frege's view this a wrong conception of meaning: He sees the meaning of an expression as its systematic contribution to the meaning of the sentences in which it appears.[5] It is thus impossible for an expression which has such a systematic contribution to be meaningless itself – this contribution is its meaning. The fact that this contribution – this meaning – is represented by an object or something else in the world that the expression refers to is an important fact with many profound philosophical implications, but recognizing (or claiming) this – intuitive and persuasive as it seems – is a separate move in Frege's theory of meaning; it is not his very concept of meaning or reference.

In BS Frege developed the theory of logical connectives ("and", "not", "or", "if-then", and the like) as truth functions, as well as the modern theory of the quantifiers

[5] This is part of the significance of the "context principle". On this see Bar-Elli (1996) ch. 5.

("all" and "there exists"). In both cases it might seem as though he showed that these expressions are "incomplete symbols" whose meaning is given by the (recursive) determination of the truth conditions of the sentences in which they can appear, but they themselves do not designate or refer to anything.[6]

At least for the mature Frege, this, however, is a fundamental error, since the very notion of an incomplete symbol cannot be reconciled with the basic principles of Frege's theory of meaning. Each of these expressions is conceived by Frege as having independent meaning which is given precisely by the above-mentioned truth conditions, since this is its systematic contribution to the meaning of the sentences in which it appears. And like all other meaningful expressions, these too refer to things in the world (first-order functions in the case of the connectives, and higher-order ones in the case of the quantifiers). It is these things that represent their meaning – their systematic contribution to the meaning of the sentences. Therefore, Russell's very idea of "incomplete symbol" must have been seen to be incoherent in Frege's view.

Not only do Frege and Russell have different concepts of meaning, but the entire system of concepts in their theories of meaning are so different that some of the key terms they use are hardly comparable. Frege has no term parallel to Russell's "proposition," while Russell uses this term ambivalently, sometimes to denote a linguistic entity – in which case the closest Fregean term is "sentence" (*Satz*) – but generally to denote a fact or state of affairs with a non-linguistic ontological status, which has no parallel in Frege's system. Moreover, Frege's system has no parallels for Russell's concepts of denotation and constituents. On the other hand, despite Russell's claim, his system has no parallel to Frege's notion of reference (*Bedeutung*), and it clearly has no

[6] See Neale (1990), pp. 1-20. Although the topic of Neale's discussion is expressions that are not "genuine referring" ones, rather than incomplete symbols, he presents the discussion in the context of a summary of Russell's theory, and Russell considered such expressions incomplete symbols.

parallel to Frege's notion of sense, which Russell explicitly and systematically rejected. It is thus clear that in order to compare the two theories it will often be necessary to use the terms at issue somewhat imprecisely.

Logical Structure in Frege

The logical structure of a sentence and the significance of attributing a particular logical structure to a sentence are associated first and foremost with the relations of implication in which the sentence is involved. The logical structure of a sentence must reflect the formal logical aspects that determine these implication relations. From this viewpoint the concepts in terms of which the logical structure is presented, and the way it is presented, are subject to the logical theory of implication or inferential relations, according to which the relations of implication in which a given sentence is involved are determined by its logical structure.

This association is at the basis of logic, but it was explicitly conceived and formulated by Frege, who was then followed by most of the logicians and philosophers of modern logic. Logic had always been considered the theory of implications among sentences, nor was the idea that sentences have a formal structure an invention of Frege's. What was new in Frege's approach to this issue is the conception of the inherent connection between the two, so that the theory of structure is no longer separate from the theory of implication relations.[7] The title of Carnap's famous book, *The Logical Syntax of Language*, can be seen as a sort of slogan expressing this awareness. Syntax deals with the (inner) structure of sentences, while logic deals with the (outer) implication relations among sentences; the Fregean idea was that they are two aspects of the same theory.

But the logical structure of a sentence (as hinted at by the connection just described) is linked not only with its implication relations with other sentences but also with the way the sentence itself is understood. The logical structure thus reflects the way

[7] See on this Bar-Elli (1985); here chapter 2.

we understand the sentence on the basis of our knowledge of the meaning of its constituents.

There are, to be sure, important mutual relations between this aspect, which involves the way a sentence and its meaning[8] are understood, and the previous aspect involving the implication relations of the sentence. But if the phrase "to be sure" is in place here, it is only thanks to Frege, who made this issue one of the basic elements of his general logical approach. For this link is far from being obvious, and there are many systems and logical calculi which seem to deal with implication relations without committing themselves to the idea that structures they impose on sentences are indeed the ones that enable us to understand the sentences on the basis of their constituents. This claim was actually one of Frege's main arguments against Boolean logic, which prevailed in his time. It was also the basis of his insistence that his logical system was a complete language in which all sentential contents could be fully expressed.[9]

But what is the nature of this link and how tight is it? Can different logical structures be ascribed to the same sentence? Are there objective reasons for preferring some of them rather than others? What are the guiding considerations here? Frege frequently notes that a sentence can be ascribed different logical structures and that sometimes it is especially convenient to do this. He nevertheless insisted that a thought has a particular structure, on the basis of which it is constructed out of its constituents, and that any sentence expressing the thought reflects this. These might seem to be contradictory claims, but it becomes easier to understand Frege's view if we make use of the distinction proposed by Dummett between decomposition and analysis (see his 1981b, Chap. 15 and the references there). For the purposes of deduction and proof it is possible, and sometimes even convenient, to decompose the content of a given sentence

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[8] Obviously a distinction must be made here between sense and reference, but on the general level we are dealing with here the principle is true of both, and so it will do no harm to use the comprehensive term "meaning."

[9] I elaborated on this in Bar-Elli (1985), here chapter 2.

in a specific logical structure which does not necessarily reflect the way the sentence is understood on the basis of its constituents – the latter constituting an analysis of the sentence. In Dummett's terms, the content of a sentence may be represented by many decompositions, but in general only one of them presents the "proper analysis" of the sentence.

This idea can be illustrated by the following simple example. The sentence

(1) Joe sees a snake.

can be decomposed as a subject-predicate sentence in which the property of seeing a snake (Sx) is ascribed to Joe (a), yielding the sentence Sa. For certain logical purposes this decomposition even seems most convenient and natural. Consider an inference like this:

(1) Joe sees a snake.	(1') Sa
(2) Anyone who sees a snake is afraid.	(2') (x)(Sx->Ax)
(3) Joe is afraid.	(3') Aa

This is clearly a valid inference, which is easy to prove through the use of these logical structures. From this standpoint it is presented as a special case of the familiar logical pattern which yields inferences such as:

(A) Joe is human.

(B) All humans are mortal.

(C) Joe is mortal.

We see that the logical structure ascribed to the sentences here constitutes a natural, convenient decomposition of the sentence for the purpose of the inference at issue. Nevertheless, as one learns in any elementary logic course, this decomposition is not the only possible one and is actually not the correct analysis of (1). We do not understand (1) on the basis of our knowledge of the meaning of "Joe" and of the "property" of seeing a snake, since the expression "sees a snake" is not understood as referring to a simple property, the way we understand "is human" or "is a snake." Understanding the expression "sees a snake" is a complicated process that we perform on the basis of

understanding the simple relation "sees" and the common noun "snake." I do not intend to go into this process in detail, since the general idea should be clear enough. It thus turns out that the logical structure derived from the "correct" analysis of (1) must be given in terms of the name "Joe," (a), the relation "x sees y" (Rxy) and the common noun "snake" (Sx). The structure we end up with is

(1") (Ex)(Sx&Rax).[10]

This result is radically different from the previous decomposition, (1'). We tried to show that (1") is actually the correct analysis of (1), through the use of concepts which allow us to understand this sort of sentence on the basis of its constituents. This is clearly a global constraint which affects not only this sentence and this inference but all sentences in which the relevant expressions appear. There are obviously many inferences and implication relations involving (1) which require a decomposition of the sort provided by (1"), while decompositions of the sort provided by (1') would not be of any help and would seem obviously wrong. Consider, for example, a case in which we want to say that Jack (b) saw the same snake that Joe saw: (Ex)(Sx&Rax&Rbx), or that Joe killed (Kxy) the snake he saw: (Ex)(Sx&Rax&Kax), or that Joe saw two snakes: (Ex,y)(Sx&Sy&Rax&Ray&~x=y). In each of these cases, which seem essentially related to the meaning of (1), we require an analysis like (1") rather than one like (1').

These considerations suggest the following important question: Is the concept of the "correct" analysis any different from that of the "ideal" decomposition – the one which is useful in all, or most inferences, and in the presentation of all the relevant implication relations? Is there such an "ideal" decomposition at all? Or do we have no choice but to make do with the use of different logical structures for different purposes, without any "ideal" or "correct" one? This is not the place to clarify the difficult issues involved in these questions. Frege, however, thought that the considerations determining

[10] Frege uses a similar example in "Introduction to Logic", PW 187/203; his example is: "Christ converted some men to his teaching".

which possible logical decomposition constitutes the "correct" analysis of a sentence do not involve merely a sort of maximization of the number of relevant inferences in which the sentence appears, but also include various semantic and epistemic considerations connected with the way the sentence is understood and learned and the way its constituents are given.[11]

An important way of looking at these semantic and epistemic considerations involves the concept of "about" – that is, Frege's realization that every meaningful sentence is about specific things, and that it is this relation between the sentence (or the thought it expresses) and the things (objects, functions) it is about that constitutes the meaning of the sentence and the way it is given to us. We will see later that this point is important in understanding Frege's opposition to Russell's theory of descriptions. Frege's basic idea here is composed of two complementary principles. The first is that a sentence is about what is mentioned in it explicitly, that is, the references of the names appearing in it. The second is that what a sentence is about serves a vital function in understanding the sentence and determining its truth or falsity.[12]

Sentence (1), for example, is about Joe, the concept of a snake and the relation "x sees y". The sentence is not about any particular snake, because no particular snake is mentioned in it (it does not contain a name of any particular snake), but the sentence **is** about the concept snake: It says that this concept is not empty, that there is something that is a snake. It also says that the concept "a snake Joe sees" is not empty, and this is actually the concept that the sentence is first and foremost about: Existential quantifiers, in Frege's view, are second-order predicates denoting second-order properties of first-order concepts.[13] Thus, in the above case we were making an assertion about the concept

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[11] See Bar-Elli (1996) chapter 8.

[12] I discuss this issue at length in Chapter 7 of Bar-Elli (1996).

[13] And not that of an object; see FA, around Sec. 56; Introduction to BL, p. 5; PW 254/274; and many other places. See also Chap. 7 of my book (see note 2).

"a snake Joe sees" and saying that it applies to something. We first form this complex predicate, and then attach the quantifier (a second-order predicate0 to it (These formulations of Frege's ideas are not entirely precise, but they should be sufficient for the present purpose.)

This view of the concept of about is of great significance for understanding major aspects of Frege's system which cannot be discussed here, but there is one point that is essential for the rest of the present discussion. This view of Frege's, especially its first principle, is an expression of what may be called the "local lexical" position on the concept of about: What a sentence is about is determined by aspects of its constituents themselves, and this is accomplished "locally", that is, irrespective of global aspects of the sentence as a whole. On Frege's view, every name and expression in the sentence determines its own reference on the basis of its own sense, and these references constitute what the sentence is about. Though, according to the "context principle" of FA, a name has a sense only in the context of a sentence, the global aspects of the sentence, such as its structure, its truth, and its relations with other sentences, play no role in determining this relation of about.

With this background we can now briefly formulate the main thrust of Frege's opposition to Russell's theory of descriptions as follows: Russell's theory claims that a descriptive sentence (one containing a definite description) is an existentially quantified sentence in which the object described in the sentence is not named or mentioned in the sentence in any way. This theory thus seems to miss the basic insight that such a sentence is about an object – that this is the way we understand it, this is the way we try to determine its truth and falsity, and the like. Since, in Russell's analysis, the object being described is not named in the sentence, it turns out that according to Frege's principles the sentence is not about this object, which appears counterintuitive and opposed to our understanding of the sentence. Instead Russell's theory interprets the sentence as an existentially quantified one – that is, as Frege sees it, a sentence about a concept.

Definite Descriptions In Frege's Logic

We have seen that Frege thought that definite descriptions in natural languages are proper names. According to a moderate version of his theory they could lack reference, and then sentences containing them would lack a truth value. On a stricter view the idea of a name having a sense but not a reference is incoherent, and Frege's willingness to accept this is a sort of conciliatory gesture of his towards natural language, which he considers a logically flawed instrument in any case. Actually, an expression lacking reference cannot be a name or a meaningful sentence, but at most a "mock name" or "mock sentence."

In a logic such as Frege's, which is constructed by assigning different functions to different levels, there are in principle two different ways to functionally create a proper name (e.g., "the present King of Jordan"):

(I) As a first-order function with an object (possibly an extension) as argument: f'(x).

(II) As a second-order function of a first-order concept: f''(Gx).

Frege was aware of both methods, but he generally preferred to use the first one. This method is very convenient for introducing e.g. arithmetic functions, as it is natural to consider them first-order functions, like the successor of 3 or the sum and 2 and 3). Frege also defines the description function in BL (§11) in the first way, as a first-order function of objects – the value ranges of first-order predicates (these value ranges are, more or less, sets or extensions of concepts). The definition is the following (the slash represents the description function and the expression x^(Fx) the extension, the range of values of the concept F): \x^F(x) = n, where n is the only object in the extension of F (i.e.the only object to which the concept F applies; in this case we also say that the expression to the left of the equality sign signifies n). In all other cases (that is, when F applies to more than one object or to no object at all), then \x^F(x) = x^F(x), that is, the value of the function in this case is the extension of the function F itself (see BL, 50). As mentioned, this is a first-order function, and in all cases the description functor is defined in such a way that any expression of which it is a constituent has (one and only one) reference.

Frege suggests that there is a great similarity between this function and definite descriptions in natural languages, and adds that the function could replace the definite article in such languages. Frege makes two important claims in support of his definition. First, he repeats his claim that definite descriptions, as understood in natural languages, can be ambiguous (e.g., "the square root of 2") or lack any reference at all (e.g., "the rational square root of 2"), which he considers grave flaws that are avoided by his proposed system. Second, his system also avoids arbitrarily fixing a reference for such a failed description. The central point he is making here is that an arbitrary determination of a reference for a definite description (or any other name) would prevent any logical connection between it and the way the name is created. The way we understand descriptions in natural language involves the principle that "the F is F" is conceived as a logical necessity, and arbitrarily fixing the reference of a failed description would flout this principle. Frege claims that the definition he presents in BL avoids this flaw.

It seems to me that this proposal of Frege's is a correction of a previous proposal of his (in SR 70/41), where he himself suggested arbitrarily fixing reference in cases of failed descriptions. This correction is extremely significant, yet it seems to have escaped Frege's commentators, beginning with Russell, who took Frege to task in OD for the arbitrary determination of reference in his system, and ending with contemporary commentators (see, e.g., Linsky, pp. 1-30). I am not claiming that Frege's corrected proposal is free of the problem of arbitrariness, but rather that he gave this problem some consideration and his theory approaches this gaol. Anyone who claims that Frege failed in this must show that this is the case with his corrected proposal. It is important here to understand that the range of values of a function is not given merely as an ordinary set given extensionally, but is defined through the function, even when it is empty. Thus, in order for Frege to determine the identity conditions of the value range, he needed to use his problematic Axiom V. This is why I used the reservation "more or less" in my above explanation of this definition of Frege's. A more detailed discussion of this issue would, however, be out of place here and must be left for another occasion.

From Frege's point of view, the best approximation of Russell's theory of description is as a theory that considers definite descriptions to be second-order functions (of type (II) above). And indeed, Russell's description operator (the iota of his PM) seems on the face of it to be a second-order functor (operating on first-order functions). But it is clear that this description is misleading, for if it were correct there would be no reason to consider definite descriptions as "incomplete symbols" that are meaningless in and of themselves. After all, the entire point of Russell's theory and proposed analysis shows that this apparent approximation from Frege's viewpoint is illusory, as the resulting expression (the description) is, in Russell's view, meaningless and certainly does not refer to anything. It therefore seems that although Russell's description operator in PM signifies a second-order function, his overall theory construes it as incomplete symbol that is analyzed away.

Here we arrive at the heart of the issue between Frege and Russell and the problem in understanding Russell's theory. According to Russell's proposed analysis of descriptive sentences, they are basically existential quantifications, and so they must be understood as being about concepts rather than objects, which is contrary to common sense and the common understanding of such sentences.

Is this indeed the case? Is it indeed correct that Russell's theory of descriptions cannot be understood in a way that would preserve the basic intuition that descriptive sentences are about objects, even though these objects are not explicitly referred to in the sentences? Is Frege right in his "local lexical" claim that the only way to understand the fact that a descriptive sentence is about the object under description is by saying that the sentence contains a constituent that refers to the object? In presenting now what seems to me to be the main philosophical significance of Russell's theory of description, I want to claim that the answer to these questions is "no" and that Russell's theory of descriptions actually suggests a revolutionary alternative to Frege's localist approach.

Logical Structure and Intentionality In Russell's Theory of Descriptions

The prime significance of Russell's theory of definite descriptions is that it constitutes an analysis and explanation of the "mechanism" of descriptive reference – the way definite descriptions enable us to think and talk about the objects they describe. It is very easy to get this wrong, and indeed many people have mistakenly believed that the entire significance of Russell's theory is the proposal to replace one way of speaking – descriptive sentences containing definite descriptions – with another – existentially quantified sentences. For example, the sentence "The King of Jordan is bald" would be replaced by the sentence "There is one and only one thing that is a King of Jordan and it is bald." This substitution, as Russell showed, solves many problems raised by the original sentence; the details are well-known and I shall not repeat them here. According to this interpretation, the most Russell's theory does is to reduce descriptive reference to an object to a "pure" quantification; I call the quantification "pure" because in the final analysis only variables and general predicates appear within the scope of the quantifiers.

A good deal of support for this idea may be found in what Russell himself said. He claimed that expressions such as "the King of Jordan" are meaningless in and of themselves ("incomplete symbols"), that King Hussein is not a "constituent" of the proposition expressed by the sentence "the present King of Jordan is bald" (of course, his examples were different), and that the substitution he proposed permits the elimination of definite descriptions as well as other denoting expressions. Yet, I believe that a deeper understanding of Russell's position presents a different picture in which his theory remains object-oriented, and constitutes an **analysis** of the descriptive way of thinking and speaking about objects. This is a philosophical analysis which points out an important conceptual connection between our ability to describe objects and our understanding of quantification – a connection which is not a reduction of one to the other or a total elimination of one in favor of the other. As I see it, Russell is basically claiming that descriptive sentences, in the ordinary sense of the term and when they are

true, are indeed **about the objects** they describe, and his theory of descriptions proposes an analysis of this idea and explains how it works.

In order to see this clearly we must discuss a number of points, which I briefly review in what follows. These points should bring to light a radically different picture from the Fregean one regarding the connection between logical structure and the about relation. Its pivot is that a descriptive sentence may be about something it does not mention or does not refer to explicitly, and which is not one of its constituents: What such a sentence is about is not determined "locally" by a particular term or by the meaning of that term, but by the logical structure of the sentence as a whole.

(i) **The status of the about relation**

Russell held the generally accepted intuitive position that every meaningful sentence is about something and that this property is an essential aspect of its meaning. Even though, to the best of my knowledge, he never formulated this principle as clearly as one could have liked, it underlies most of the work he did at that time in the areas under discussion. In *The Principles of Mathematics* of 1903 (PoM), for example, he distinguished between two categories that constitute what he called a "proposition" (which he considered at that time an ontological rather than a linguistic concept, even though he was not careful about this distinction and was rightly blamed for that): a term and an assertion. He claimed that every proposition contains an assertion about some term (or several terms, if a relation is involved; see, e.g., pp. 39, 44). Within this general conviction Russell introduces the denotation relation as a relation that is meant to explain how a proposition can be about things that are not its direct constituents. His explanation relies on a special logical relation between a denoting concept and the object it denotes. Moreover, a denoting concept is defined in the book as a concept contained in a proposition as a constituent, such that the proposition is not about it but about the object it denotes, by means of this special logical relation (p. 53).

It is simply impossible to understand the issues under discussion and the concepts involved in them without considering the central status of the problem of intentionality

and the concept of about. In PoM Russell did admit the possibility of "empty descriptions" (that is, denoting concepts such that there is no object they denote), but what he said on the subject was brief and amazingly obscure. What are propositions that contain such concepts about? They cannot be about the concepts themselves, by the very definition of a denoting concept, nor about the object they denote, since no such object exists. Thus it turns out that such propositions are not about anything, a notion that is hard to fit into Russell's system of ideas at that time. These remarks are especially apt in light of the fact that in PoM Russell conceived the denotation relation as a primitive logical relation. How can we reconcile the idea that a denoting concept has an independent, essential, primitive logical relation to an object with the idea that the existence of such an object is a contingent empirical matter, and even if the object does not exist the denoting concept retains its force as a constituent of the proposition in which it appears?[14]

This thorny issue is resolved in the 1905 OD theory of descriptions, in which the denoting concepts of PoM disappear entirely, along with the mysterious logical relation between them and the objects they denote. The new theory allowed Russell to hold the about principle without qualification: Every proposition is about somethings – its constituents, which it always has – and a descriptive proposition is **also** about the object its description denotes, when such an object exists. (The significance of this issue will be discussed later.[15]) Thus according to this view there cannot be a case in which a proposition is not about anything.

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[14] These two points seem to have escaped Hylton (1992), who claims that Russell's theory of descriptions in PoM provides a plausible explanation for the problem of empty descriptions, even though Russell himself probably did not see it this way at the time. See pp. 73, 427.

[15] This is in contrast to Neale's (1990) view that the fact that descriptions are not denoting expressions actually implies taht descriptive propositions can never be about the object denoted by a description, if such an object exists. See Neale (1990), pp. 2-21

(ii) **The object-directedness of descriptive propositions**

Descriptive propositions are not merely "about something," they are typically about objects – those they describe. "Typically" here means that this is how descriptive propositions are ordinarily grasped in the paradigmatic cases that constitute the basis for understanding them. It does not mean that a descriptive proposition cannot fail in that the object it describes does not exist, but that the possibility of such a failure, and the fact that it is then grasped as a failure, do not undermine the fact that descriptive propositions are typically and paradigmatically grasped as referring to the object they describe and as being about this object. This view is opposed to most interpretations (sometimes called "universalistic") of Russell's theory, which claim that the quantificational analysis of descriptive propositions makes it necessary not to conceive them as directed at, or being about an object, and that such a conception is definitely not vital for understanding them.[16] But these interpretations seem to me to miss the point of Russell's intention and the significance of his theory as the analysis of an essential aspect of our referential ability.

I shall mention three points in which Russell expresses the conception of descriptive propositions as directed at objects. First, Russell's theory, as the title of the paper indicates, is about denoting. For Russell the problem of denoting is the question of how a proposition or thought (or a sentence) can be about objects or other things in the world. The main purpose of the theory was thus to discover and explain the concepts and modes that are involved in this relation and make it possible. The problem that occupied Russell in OD was first and foremost the possibility of explaining **successful** descriptive reference: How a descriptive proposition can be about or refer to objects that are not its constituents. The universalistic interpretation claims that descriptive propositions are actually (in contrast to their surface appearance) not about objects at all. If this were correct as a general interpretation of Russell's theory, and not merely in some particular

[16] See, e.g., Linsky (1977); Blackburn (1984), p. 310; Neale (1990), Chap. 2

cases, then it would remove the basis of the very problem the theory was designed to resolve. True, the universalistic interpretation relies on an important aspect of Russell's theory, which enabled him to explain the meaning of sentences containing empty descriptions. I do not deny this important aspect, but excessive focus on it is liable to distract our attention from the primary significance of the theory: explaining the possibility of successful descriptive reference, namely, our ability to think about objects by means of descriptions. Recall that in OD, after presenting his theory, Russell defines the "denotation" of a descriptive sentence. This would hardly have a point if he really meant to get rid of the concept of denotation and show that it is merely a superficial feature of the surface structure of sentences – that descriptive sentences are not actually directed at any denotation but are just ordinary quantified sentences.

Second, it is important to look carefully at the way Russell himself presents his primary interest and the philosophical motivation of his theory. This is how he presents it near the beginning of OD:

> "We know that the centre of mass of the solar system at a definite instant is some definite point, and we can affirm a number of propositions about it; but we have no immediate acquaintance with this point, which is only known to us by description. The distinction between acquaintance and knowledge about is the distinction between things we have presentations of, and the things we only reach by means of denoting phrases" (LK p. 41).

Russell does not speak in terms of "denoting concepts" here, as he did in PoM, but in terms of "denoting phrases"; he nevertheless remains faithful to an important element of the old notion: that it is by means of these denoting phrases that we reach the things themselves, even though they are not the constituents of the proposition. A little later in the paper Russell writes:

> We do not necessarily have acquaintance with the objects denoted by phrases composed of words with whose meaning we are acquainted. ... hence, what we know about [other people's minds] is obtained through

> denoting. All thinking has to start from acquaintance; but it succeeds in thinking about many things with which we have no acquaintance (p. 42).

This ability of our thinking is what Russell was trying to explain in OD – the success of our descriptive ability in referring to the objects being described.

Third, a central epistemic aspect of this issue became the basis of Russell's epistemology: the distinction between knowledge by acquaintance and knowledge by description. To Russell, the concept of knowledge by description is that of knowledge of objects: What we know "by description" is the described object, despite the fact that our knowledge is analyzed within a theory of descriptions which proposes a translation of descriptive sentences into existentially quantified ones. The theory of descriptions itself thus should not be understood in a way that eliminates the objectual nature of the description and the knowledge it contains. A fuller discussion of this issue would not be in place here; I have discussed it and its significance elsewhere.[17]

(iii) **"Remote" intentionality**

Earlier we discussed the Fregean principle that a sentence is only about what is mentioned in it explicitly – in other words, what constitutes the reference of the names appearing in it. If, for convenience, we call these the constituents of the sentence, then the principle says that a sentence can only be about its constituents. An important aspect of Russell's position, according to the interpretation proposed here, is that it rejected this principle. A descriptive sentence (or proposition – the distinction is not important here) can be about an object "external" to it – an object that is not one of its constituents. To see this we need only recall that a descriptive sentence never contains the object it denotes, that is, the one it describes, as one of its direct constituents; but in the paradigmatic case, when it does denote an object, it is about this object. Not only does

[17] See my paper: "Acquaintance, Knowledge and Description in Russell", *Russell*, vol 9\2, 1990, 133-157

Russell's position here differ from Frege's, but I cannot think of a theory preceding OD which explicitly expresses the position that a sentence (or a proposition) can be about an external object, which is not one of its constituents.

In PoM Russell's position on this issue is unclear. Many people have thought that in PoM too the object denoted by the denoting concept is external to the proposition and not a constituent of it. Hylton, for example, insisted on this point, turning it into one of the basic elements of Russell's system in PoM, since it allows us to grasp and think about infinite totalities, which, as such, are not constituents of finite propositions. But this seems to me a very doubtful claim. To the best of my knowledge Russell never said this anywhere in PoM, and there are a number of places where he said the opposite. For instance, concerning the proposition "Socrates is a man," he says that it has three **constituents** – a term (Socrates), a relation (apparently predication), and an object (he calls "disjunction") that is denoted by the denoting concept "a man" (p. 54; see also pp. 44, 46, 47). This means that the "disjunction", which may be infinite, and is a denoted object here, is indeed contained in the proposition as a constituent. So we must say at the least that Russell may not have been entirely clear on this point in PoM. Indeed, this may have been one of his reasons for abandoning this theory and developing the theory of descriptions in OD in its place.

(iv) **Intentionality and logical structure**

The last point, which is perhaps the most important one, is that Russell's theory of descriptions presents the position that the denotation of a sentence – what the sentence is about, when it has a denotation – is not determined solely by the "local" aspect of one or another of the constituents in the proposition, but by the logical structure of the proposition as a whole. This issue too constitutes a revolutionary turn from Frege's position, from that of PoM, and actually from all other previous conceptions of reference that I am aware of. In Frege's view, as noted before, the reference of an expressions, and what a sentence containing it is about, are determined locally, lexically, by an aspect of the expression itself, namely, its sense. Once this sense is given, global aspects of the

sentence as a whole, such as its logical structure, do not play any role in this. In Russell's theory of descriptions, by contrast, the denotation – what the sentence is about – is determined by the logical structure of the sentence as a whole. The logical structure thus gains major importance in determining the intentional structure of sentences, as well as our ability to refer and to direct our thoughts at objects and things in the world about which we think, know and speak.

It is in light of this that my earlier remarks on the significance of Russell's theory of descriptions should be understood: It is a theory that analyzes our ability for descriptive reference in terms of the theory of quantification and the logical structure of quantified sentences. The procedure proposed by Russell might have appeared as a theory that reduces descriptive reference to quantification in such a way as to present the very concept of descriptive reference as redundant and misleading. But even if this were possible, it seems not to be the way Russell himself perceived the issue. As far as the motives and deep philosophical significance underlying the theory of descriptions as Russell perceived it can be reconstructed, the theory does not eliminate the concept of descriptive reference and does not prove it to be superfluous; on the contrary, it insists on the major significance of this concept, while explaining and analyzing it in terms of the theory of quantification. Russell's theory of descriptions, in this view, does not determine a one-way reduction of descriptive reference to quantification, but (on the assumption that it is correct) reveals a mutual conceptual connection between them.

At the outset I asserted that the debate between Frege and Russell on definite descriptions touches upon fundamental problems in philosophy, especially the nature of the connection between the logical and the intentional structure of sentences. We saw that Frege was in possession of all the insights about definite descriptions and all the logical tools needed for constructing a theory similar to Russell's. From this standpoint, and from a "pure" logical perspective, Russell's theory is not great news over the ideas Frege had already developed on this issue. Moreover, we have shown that Russell's theory is somewhat unclear and suggests a view whose very coherence is debatable: It exists

within a rather vague, narrow range between perceiving descriptive propositions as directed at objects, on the one hand, and as "universalist" quantifications, on the other; between perceiving them as being about objects or as only being about concepts and functions; between considering the theory as a proposal for a reductive elimination or as a conceptual explanation and analysis. Perhaps it was also Frege's pedantry on clarity that prevented him from accepting such an ambiguous view. But although Russell's theory, in my interpretation, is indeed somewhat ambiguous, I believe that this ambiguity is a fruitful one that expresses the mutual link between our ability for descriptive reference and our understanding of quantification. But, in addition, the theory of descriptions, properly understood, undermines some of Frege's fundamental views about the intentional structure of propositions and thoughts and how it is determined. It is on these issues that Russell's theory of descriptions was most radically and significantly innovative – its idea of "remote intentionality" and of the determination of what a sentence is about by virtue of the logical structure of the sentence as a whole.

This innovation of Russell's has far-reaching philosophical significance, on which we cannot expand. Here I briefly mention two of its major aspects:

1. This view fits the radical philosophical significance of Frege's "context principle" and the primacy of whole sentences in explicating the basic concepts of a theory of meaning. According to these principles the meaning of an expression is conceived in terms of its contribution to the meaning of the sentence, in which it appears, as a whole. The logical structure of a sentence is a global, propositional property. The view that what a sentence is about is determined by such a global propositional property fits squarely into a general picture of whole sentences and propositional properties as the basis for clarifying fundamental concepts in the theory of meaning.

2. The view under discussion also makes it possible to explain the problems that arise due to the existence of empty names and descriptions more convincingly than can be done with alternative views.

(a) We have seen (pace Hylton) that the problem of empty descriptions is not satisfactorily resolved in PoM. There Russell held three claims that are difficult to reconcile. One is that the question of whether there is an object that a descriptive proposition is about is determined by the mysterious "special relation" between the denoting concept appearing in the proposition and the object it denotes. The second is that this relation is a primitive logical one, which is never explained within Russell's theory in PoM, but is presented there as one of the primitive bases of logic. The third claim is that sometimes a denoting concept does not denote anything, because there is no object for it to denote, and this is generally a contingent empirical fact.

It is hard to see how these three claims can be defended together, especially the last two: If the relation between a denoting concept and the object it denotes is a primitive logical one, how can its existence in a given case be contingent or empirical?

(b) This problem is also a serious one for Frege, as we have seen. That there are names which have sense but no reference seemed to Frege a flaw in natural language, which is unlikely to have a coherent explanation, at least if one wants to use the name seriously in making assertions. In a logical language, Frege insists, every meaningful expression has a reference, as long as it functions in a genuine statement, expressing a genuine thought, which must be either true or false. However, says Frege, it is not always genuine statements or thoughts that we are interested in; sometimes we are concerned with mock statements or mock thoughts (cf. PW 141-42/130). Indeed, whenever Frege speaks about names lacking reference he is always careful to point out that these involve poetic or fictional contexts – in general, art rather than science. Frege's attitude towards art was that it is essentially subjective, while he considered genuine thoughts and statements essentially objective. Therefore, in his view, thoughts and statements in artistic contexts are not "genuine", with an objective status and a truth value, but are at most something resembling genuine thoughts and statements – mock thoughts and mock-statements .

(c) All this is entirely different in Russell's theory of descriptions, where the object denoted by a description or what a descriptive sentence is about is determined by the logical structure of the sentence as a whole rather than by any local, lexical factor. Such a position is not and need not be committed to a view in which the meaning of any expression (whether a definite description or some other denotative expression) or what a sentence containing that expression is about, depends on the existence of the described (or denoted) object, since whereas the constituents of the logical structure are meaningful (and have a reference) by virtue of the existence of the things that constitute their meaning, the logical structure itself can fix upon an object as the thing that the sentence is about, where this possibility does not dependent on the existence of the object. From this standpoint the logical structure is a propositional property of the sentence as a whole, and it is grasped and understood as such whether or not it "succeeds" in determining the denotation of the description (in case there is an object that satisfies the description), just as it is understood this way whether or not the sentence is true. The logical structure of a sentence is an essential factor in determining its truth or falsity, but it is understood as such independently of whether or not the sentence is actually true. Likewise it is understood independently of whether or not it has a denotation.

This rejection of previous conceptions that what sentences are about is determined solely by local, lexical factors, and its replacement by a new theory of descriptions in which it is determined by the logical structure of the sentence as a whole, are great innovations in themselves, but they also pave the way for the new approach Russell suggests for understanding sentences containing empty expressions.

I conclude with two somewhat "paradoxical" remarks about the historical status of Russell's theory of descriptions, which I present briefly as programmatic suggestions which require a separate discussion. One was hinted at above, namely, that it is precisely the major aspect of Russell's theory which I discussed mostly in its anti-Fregean aspect – determining reference by virtue of logical structure, which is a global property of the sentence as a whole – that fits the constitutive principle of Frege's entire theory of

meaning: the context principle. For, after all, the main point of the context principle, and of the primacy of sentences derived from it – namely, that the meaning of any expression is its contribution to the meaning of the sentences in which it appears – can be interpreted, by means of a suitable generalization, as the requirement that all the basic concepts of the theory of meaning, including the concept of about and the determination of reference, should be based on propositional concepts, such as the logical structure of the sentence as a whole. Elsewhere I have tried to show that the key concepts of Frege's theory of reference and the constitutive constraints on his concept of sense can and should be grounded in the concept of about. I have also pointed out that one of the advantages of this approach is the propositional nature of the concept of about, as it is the sentence as a whole that is about an object.[18] From this viewpoint Russell's theory of descriptions as I have presented it here constitutes a crucial additional step in this direction: Reference itself (at least of the descriptive sort) is determined by virtue of a clearly propositional property – the logical structure of the sentence as a whole. This touches on one of the greatest and deepest foci of tension in all of Frege's philosophy: the tension between the local nature of his theory of sense (since every expression has a sense that determines its reference), on the one hand, and the context principle, together with the primacy of the sentence, on the other (I am not suggesting that it is irresolvable).

My second closing remark is about Kant. From many aspects Russell was an anti-Kantian philosopher who worked hard to revive classic British empiricism. If I am right, however, in my suggestions about the connection Russell established between the logical and the intentional structure of sentences, and about his revolutionary innovation with respect to Frege's view, then Russell's theory of descriptions could actually be interpreted as a revival and weighty explication of one of Kant's more obscure ideas: that objective

[18] See chapters 1and 7 in my book, op.cit., note 2.

reference – the way thought is directed at objects in the world – is a function of the categories of logic. It is very difficult to understand what exactly Kant meant by this obscure doctrine, which has been offered many interpretations, and Kant certainly did not think of the logic Frege and Russell developed. But Russell's view as presented here actually supports this general doctrine and explains it: Descriptive reference, as a case of reference to objects, is determined by virtue of the fundamental concepts of the theory of quantification and the logical structure of quantified propositions. It is precisely Frege, who was in many respects an outstanding Kantian philosopher, who seems much further from Kant than Russell is on this issue.

Chapter 6: On the Ontological Status of Senses (Sinne) in Frege

Ontology is the general metaphysical discipline of what there is. General, because on a more specific level questions of existence pertain to specific sciences: the question of whether there is a biggest prime belongs to mathematics and the question of whether there are passages of energy faster than light, or dark matter belongs to physics, etc. On the general level of ontology it doesn't deal with these questions but with questions e.g. about the nature of reality – whether for instance it is mental or material. On a somewhat less general level ontology deals with categories of reality and of what there is – which if any are the more fundamental, which can be reduced to others or supervene on others etc. Evidently another prime question of ontology is how we decide about these matters, and what kinds of argument we can bring for them. Though ontology is one of the oldest branches of metaphysics, focusing on this last question is characteristic of modern ontological enquiries. And within this broad field, characteristic of modern analytic ontology is the view that philosophy of language has here a primary role.

Although almost any great philosopher had interesting things to say about language and meaning, philosophy of language in the modern sense begins with Frege, and the gist of his philosophy of language is his theory of reference (*Bedeutung*) and sense (*Sinn*). As a consequence of his investigations in the philosophy of language Frege revolutionized ontology, as we shall see, in two main respects: He established the realm of references as consisting of two exclusive categories – objects and functions – undermining other categories and distinctions that dominated previous ontological research, as between particular and universal, concrete and abstract. In addition he established the complementary objectivity of senses as consisting of sui generis entities distinct from both objects and functions. This last point will be the focus of our discussion.

Let me begin by some general remarks on Frege's theory of meaning, which is a necessary premise for what follows. I shall be very brief, for in general it is well known and many expositions of it are available. I shall therefore also be very sparse in references to Frege's writings in these remarks. Some aspects of the theory are complicated and technical but its main idea can be put in three simple sentences:

1. Meaning is ascribed primarily to complete declarative sentences (or propositions).
2. The meaning of a sentence consists mainly in its truth conditions − the conditions in the world in which it is true and those in which it is false.

3. The meaning of sub-sentential expressions (like names and predicates) consists in their systematic contribution to the meaning of sentences of which they are constituents. (It is in spelling out the "systematic contribution" that the technical complications enter, and I won't go into them).

I phrased the above in terms of "meaning" but it applies to both reference and sense. The general idea of the reference of an expression is that it determines the truth or falsity of sentences in which it is a constituent.[1] Thus the truth of "Jerusalem is hilly" is determined by the reference of the name "Jerusalem" and of the predicate "...is hilly". Naturally, the former is taken to be the city of Jerusalem, and the latter is the property of being hilly, and this is so primarily because the sentence is about Jerusalem and this property.[2] This may look so simple and intuitive that it is easy to miss its point. And the point is a conception of a general **theory** designed to apply to all words in all meaningful sentences. Not only did Frege see the need for this but he in fact spelled out the outlines of such a theory. He did it in the framework of a logical language he called *Begriffsschrift* − conceptual notation – which is marked by another ingenious idea of his, namely that the inner logical structure of a sentence is what determines and accounts for its (outer) implication relations – what is implied by it (perhaps with other sentences) and what it is implied by.[3]

Already in *Begriffsschrift* of 1879 (section 8) Frege realized that for understanding a sentence, reference (or what he called then content, *Inhalt*) is not enough. For such an understanding one need also grasp a particular way in which this content is given, as expressed by the sentence and its terms. Thus for instance, in understanding an identity like 24+33=57 it is not enough to know that the two sides of the identity have the same content or refer to the same thing, for then it would seem that we grasp the same in

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1 Frege's "Bedeutung" has been translated by various English terms. Most common are "reference" and "meaning". I shall use the former, leaving the latter for a more general notion insensitive to the sense/reference distinction.

2 I argued in detail for the centrality of the notion of about in understanding Frege's notion of reference in Bar-Elli (1996), particularly ch. 7.

3 This was absorbed in modern logic to the point of being almost unnoticed. But it was a novel and distinctive mark of Frege's conception of logic. I expanded on this in Bar-Elli (1985).

57=57. But surely the whole point of the identity is that this number is given in different ways by the two sides of the identity. Same is true of e.g. "Jerusalem is the capital of Israel", "the morning star is the evening star" and in fact of any identity which is not a priori trivial like a=a. Frege concluded that in a theory of meaning we need to take into account not only the reference but also the way it is conceived or given to us, as this is expressed by the term concerned.[4]

For some reasons, in *Begriffsschrift* Frege confined this observation to identities, but he soon realized that it applies to any sentence – the meaning of a sentence (that which we need to grasp in understanding it) involves not only the references of its terms but also the particular ways these terms express the manners in which the references are given to us. The same number is given differently by "57" and by "24+33" and the same city is given differently by "Jerusalem" and "The capital of Israel". This of course applies to anything: whenever we refer to something and talk (or think) about it, it is given and conceived by us in a particular way, expressed by the terms with which we refer to it. This observation is noticeable already in his FA of 1884 but found clear expression in his writings of the early 1890s, particularly in SR of 1892, in which Frege also introduced a systematic terminological distinction between *Bedeutung* (reference, also often translated "meaning") and *Sinn* (sense)[5]. A sense then is a way the reference of an expression is given to us as this is expressed by the expression (regarding indexicals , elements of the outer context are also included in their senses (T, in FR 332/64-5). This explanation is of course not a definition, so one need not be deterred by its implicit circularity. Also, the above characterization of sense as involving appeal to linguistic meaning does not imply, according to Frege, the ontological dependence of the existence of senses on language or on human institution. No, their existence does not depend on any such activity or institution, but their discovery and our knowing and realizing them do. How exactly and what is the nature of this dependence is unclear. Frege often says that our having (mental) ideas and representations, though a psychological fact, which at least partly result from our interactions with the outer world, is a prerequisite for our ability to grasp senses

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4 This conception of BS and its relationship to the later SR is discussed in detail in Bar-Elli (2006).

5 This found clear expression already in Frege's letter to Husserl, May 1891 (PMC, 61; FR 149).

(thoughts included). His theory also implies, though he was less explicit on this, that this ability is conditioned on language and on our being linguistic creatures. But these are more hints than a real account.

Frege's views of the notions of truth and about were realistic: a proposition is true or false according to the state of affairs in the world, independently of whether or how we know it. And what it is about are real things in the world. Therefore, in spelling out the theory of reference somewhat below this very general and programmatic level, he had to specify to what ontological categories references of various terms belong. For good reasons the principle Frege adopted here is that there must be a match or parallelism between the linguistic or logical categories of terms in sentences and the ontological categories of things they refer to. This is an aspect of the methodological Fregean revolution mentioned above in the second paragraph. This match is however partial: Expressions have structure and complexity and can be analyzed to their constituent parts, while their corresponding references, though **determined** by the references of these parts, have no structure and are not complexes of these parts. The reference of "The capital of Israel" is determined by the reference of "Israel", but Israel is not a part of Jerusalem. Complexity and analysis pertain not to references but to senses (more on this later on).[6]

All terms in his *Begriffsschrift* are either names or predicates (including functional terms). Predicates are distinguished from names in being incomplete, i.e. in having some empty places, which when filled by names result in another complete name (including a sentence, which Frege regarded as a name of a truth-value). References are categorized accordingly to **objects** (references of names) and **functions** (references of predicates), where objects are complete self-sustained identifiable entities, and functions are essentially incomplete (more on this later). A sentence like Fa is true if and only if the object which is the reference of "a" falls under the function (concept), which is the reference of "F" (more idiomatically put, if the function rightly applies to it). Reference here is regular or direct reference, to be distinguished from what Frege called "indirect reference" (*ungerade Bedeutung*) in oblique contexts (more on it in the sequel). The basic Fregean ontology at the realm of reference consists therefore of objects and functions

[6] This point has been widely misunderstood. See for instance Kluge, E-H. W. (1980), large parts of whose discussion, in my mind, suffers from it.

(concepts and relations being special functions), which (at first level) are functions on objects.

In order for the theory to be complete, reference must be ascribed also to complete sentences. Indeed, the main point of Frege's celebrated article SR was to argue for this and to claim (with good, though perhaps not entirely convincing reasons) that the reference of a sentence is its truth value – the True or the False – where all true sentences refer to the True, and all false ones to the False.[7] Frege went further to claim (less convincingly) that sentences are names and that their references (the two truth-values) are objects – logical objects, which are actually referred to by any thought and by any genuine statement.

Objects, according to Frege are complete, self-sustained and self-subsisting identifiable entities, while functions lack all these properties. They are essentially incomplete, essentially in need of objects to apply to. This "need" is a logical one: The whole being of a function (concepts properties and relations included) is in its applicability to objects. So, while objects like Socrates and his wisdom may depend on various things to exist, the dependency is physical and in general causal; while the property or concept of being wise depends on objects to apply to, and the dependency here is logical – the whole being of this concept is in its applicability to objects, which is reflected in the incompleteness of the expressions referring to it like "...is wise". Frege called this form of logical incompleteness unsaturatedness (*Ungesaettigkeit*), and said that predicates, as well as their referred-to functions are unsaturated[8]. There is in Frege another kind of dependency, which is sort of in between these two. An idea, an image, a feeling, an act of thinking etc. all these, which comprise the domain of the subjective, are essentially dependent on an individual's consciousness. The dependency here is not

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[7] Although this view, and its formal kernel – the celebrated "slingshot argument" – has been widely accepted, and in fact absorbed into many semantic and logical theories, it also has its opponents (see e.g. Barwise, J. and Perry, J. (1984). I shall not go into it.

[8] See, for instance "Function and Concept" (FC), 133/6-135/10 in FR.

strictly logical like in that of functions on objects, but is stronger than a mere causal dependence like that of Socrates on his parents or on oxygen.[9]

A distinctive characteristic of this view is that the ontological categories are implied and determined by the linguistic-logical categories. This is a general revolutionary view that initiated analytical ontology and in fact analytic philosophy in general. As a basic metaphysical and methodological principle it is independent of the details of Frege's logic and of his specific ontology of objects and functions. For deep reasons that pertain to the essentials of his logic Frege regarded the logical categories of names and predicates asymmetrically: though both categories are essential, there is a sense in which names form the basic category and predicates are conceived in terms of names; moreover, functional terms divide into levels, while the category of names is homogenous – all names are treated alike; there is no logical distinction between e.g. concrete and abstract, between universal and particular. The criteria for the categories are purely logical, and from this perspective these distinctions don't matter. Thus, saying that 5 is prime, that Jerusalem is hilly and that wisdom is the goal of education all ascribe a property to an object. The venerable distinctions between the one being abstract, the other concrete, the one particular and the other universal etc. don't get into the logical picture, and hence, in light of the above principle, are devoid of ontological or metaphysical significance.[10]

However, the distinction between names and predicates is a categorical logical distinction, and hence the distinction between objects and concepts (or functions in general) is sharp and categorical: no object is a concept and no function is an object. Objects and Functions have, as we have seen, completely different modes of existence and are two categorically different ontological categories. But this, according to Frege, does not deprive functions a bit of their objectivity and reality: they are real, objective entities in the world, though not objects.[11]

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9 Frege was never tired of emphasizing this, which is one of the topics most elaborated in his writings, at least since FA. For a late discussion see T.

10 I discussed some aspects of the basic status of the category of objects in Frege's view, in comparison to Kant's, in Bar-Elli (2014).

11 "No entity without identity" is a slogan often ascribed to Quine. I don't know of any place in which Quine actually wrote it, but as far as I know he also has never denied it. Since identity is for Frege a relation between objects, he would have definitely

Senses Are Real and Objective

On the background of these general principles, what should we say about the ontological status of sense? Within the framework of the Fregean ontology described so far senses should be either objects or functions – there just isn't anything else. This indeed is a prevalent view. I propose however that this is a mistake – that senses are required by the Fregean principles to be real objective entities, which are neither objects nor functions. One has therefore to allow in the Fregean ontology a special *sui generis* category of senses (including thoughts, which are the senses of propositions). We shall turn now to these two claims – that senses are real and yet not objects or functions – in more detail.

That senses are real and objective is one of the cornerstones of Frege's mature and late philosophy. There is hardly a topic that is more repeated and argued for. But what exactly does this mean? Frege distinguished being **real** from being actual (*Wirklich*). Everything actual is real but much of the real is not actual. What is actual belongs to the natural causal network and to what impacts our senses. Thus, abstract objects like numbers are not actual– they do not impact our senses and are not participants in the natural course of causes and results. And yet they are real. In what then does their reality consist? It consists mainly in their being references of constituents of true propositions. That is, they and their relations are the things in the world that "make" some propositions true. Their reality derives from that of truth, and if we hold, like Frege, a realistic notion of truth, we must regard them to be real.

And so are also senses. Thoughts, being senses of propositions and being built out of senses of their constituents, are the primary bearers of truth and falsity and are essentially true or false. They are, moreover, references of terms in e.g. belief-contexts, or generally what Frege called "oblique contexts" (more on this later) which make propositions in such contexts true or false. Hence, there are good reasons to regard them as real.

Moreover, they are **objective**, and whatever is objective is real. The notion of objectivity is quite intricate in Frege. Naturally it is introduced in contrast to subjectivity. But what is subjective, like images, ideas, feelings etc. is, though not objective, still real. Frege did not deny the reality of the subjective. He did not deny even the scientific status

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rejected the slogan. There are, in his ontology, many entities without identity – definitely functions; but I think also senses.

of psychology of the subjective.[12] What then is the objective and in what does it consist if the subjective, being real, is not objective? What is objective, for Frege, is what is real, subject to laws, and does not essentially depend on an individual consciousness, which means that it is what is in principle communicable and inter-subjective. And although Frege did not deny the reality of the subjective and the psychological, he sharply distinguished between the psychological and the logical, and between the subjective and the objective (FA, Introduction). And these for him are parallel distinctions.

The import of this brings us to what is perhaps the main characteristic of the objective – its being **rational**, its belonging to the space of reasons and justifications. This is of course connected to its communicability, but it goes much beyond it and is much more specific. Logic is for Frege the science of justification – of giving rational reasons for holding a truth. Whatever belongs to that, either in being what is justified or in belonging to the justification, is objective.[13] Since in rational justifications we deal with what we regard as truths, what is objective is by its nature real. But the other direction is. as we have seen invalid – there are real things which are not objective. Images and subjective ideas do not belong to the space of reasons, to the rational game of justifications of truths. But Frege did not deny their reality in being references of some constituents of some true propositions. This is evident in many of his writings but comes to the fore in *Der Gedanke* (T) of 1918 in which these subjective ideas, images etc. are assigned to the "second world" (the first comprising physical actual objects and the third – thoughts and senses in general). In the course of arguing there that I, the self, is an object of conscious awareness, but not a subjective idea, Frege establishes that "**not only a thing but also an idea may be a common object of thinking for people who do not have the idea**" (340/73 in FR).

Here, incidentally, Frege clearly dissociates being an object from being objective, for an idea is definitely not objective (we shall come back to that later). Senses are the locus of objectivity: reasons, justifications and truths are or pertain to thoughts, and thoughts are senses of sentences. They comprise and are built out of senses of predicates and of names. So all these are real and objective.

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12 This is argued in detail in Bar-Elli (1996); see in particular 3.A.

13 I expanded on the centrality of the notion of justification in Frege, particularly in relation to the notions of analyticity and apriority in Bar-Elli (2010).

We have seen before that one of Frege's revolutions in metaphysics was to conceive of ontology and the ontological categories in terms of the requirements of logic and a theory of meaning in the philosophy of language. An important outcome was that ontology consists of objects and functions, which are categorically distinct, but are both real and objective. We have also seen that he came to the conclusion that senses – ways things are given to us as the references of terms in propositions – are thus required in logic and in a theory of meaning. So reality, according to him, must consist not only of the references of terms of true propositions, but also of their senses.

What then is the ontological status of senses and to which ontological categories they belong? In answering this we shall first argue that senses are neither objects nor functions.

Senses Are Not Objects

Frege never says that senses are objects, and yet the view that senses in his theory are objects is a very widespread one.[14]

In discussing the matter we must distinguish at least two main senses of "object". One is a purely grammatical notion mainly of the accusative mode being the object of a transitive verb or of a relation. In this sense we can say also of a concept or a function that they are the "object" of an assertion. The other is the full ontological notion of an object, which Frege characterized as a real, independent, self-sustained and identifiable entity, which as the reference of a proper name is what a proposition containing this proper name is about, and what determines it as true or false. We shall call it the **substantive notion of object**. It is of course this which is at stake here (I shall say something about the former in relation to "oblique contexts" later). If Frege thought that senses were objects in this substantive sense, it would be quite remarkable that he had never said so. Why then people think that he thought so? As said before, the view is quite prevalent. In the Introduction to FR, for instance, M. Beaney presents this view as evident

[14] See for instance Bell (1979) ch. 4 and pp. 73-4; Dummett (1991a), ch. 12, and Dummett (1981b), pp 341-2, 504, Beaney's Introduction to (FR).

without even feeling it deserves argument.[15] But here is an argument he proposed later in another place:[16]

> "If Frege did not conceive senses as objects, then, equally, he cannot have conceived numbers as objects; but this is plainly wrong […] Such an interpretation is flatly inconsistent with one of the central theses of Frege's work – that numbers are objects […]. For even if we allow that the context principle plays a role throughout Frege's work […] that senses too are contextually defined does not imply that Frege did not regard senses as objects" (161).

This is Beaney's argument (the only one there!), which is supposed to show that senses are objects. I must admit right away that on the face of it this argument is so glaringly false, and displays such a misunderstanding of Frege's basic positions that I am afraid I am missing something exceptionally deep here. Numbers, for Frege, are objects – they are extensions of certain second-order concepts. They belong to the realm of reference and are **definitely not senses**. Why then does Beaney think that if numbers are objects so must senses be (or as he puts it, that if senses are not objects so are numbers)?! He owes us some explanation for this quite fantastic claim, but he gives none. It should be added, by the way, that even if we grant, for the sake of argument, that some senses were objects (or should be so regarded for some purposes), this would still fall short of showing that senses in general (thoughts included) were. Evidently, as emphasized above, senses are objective, and in denying as we do that senses were objects we still need to explain in what their objectivity consists. I have tried to do something in this direction elsewhere and sketched the main points above, emphasizing in particular their inter-subjectivity, communicability and their role in justification.

The remarks about contextual definitions (only partially quoted here) at the end of the above quote from Beaney are not only inaccurate in themselves (mainly in talking of contextual definitions of **senses**, and in blurring the difference between the context principle and contextual definitions), but utterly irrelevant to the question of whether

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[15] E.g. 18. There are many versions, with some important differences. I shall concentrate here on Beaney because his Introduction to FR is widely read.

[16] Beaney, M.: "Review of Bar-Elli", *Mind*, January 2001.

senses are objects (in the substantive sense). There may be room to wonder whether the fact that, say, directions can be contextually defined (i.e. the truth conditions of e.g. identity between directions can be specified in other terms) gives one sufficient reason to deny that directions are objects. I don't think so, but in any case this has nothing to do with the question of whether **senses** (including senses of directions) are objects.

It is not only that Beaney's "argument" is worthless; the view he tries to defend – that senses are objects – is very problematic, and may be based on misunderstanding one of Frege's main philosophical strands. Let me explain in brief the main issue. As stated above there is no question about the objectivity of senses in Frege's philosophy. This is indisputable. Does this imply that senses are objects? There is a widespread tendency to associate the notions of objectivity and object: Objects are objective, and what is objective, one might suppose, must have the status of an object ("the objecthood of the objective" we might call it).[17] But this tendency is philosophically muddled and opposed to Fregean theses, which are amongst his philosophically most important ones. We have seen before that e.g. in *Der Gedanke*, Frege dissociates being objective from being an object. Concepts and Functions are objective – this is a cornerstone in Frege's philosophy. But they are definitely and categorically not objects. Hence, even within the realm of reference, Frege dissociates the notions of object and of objectivity so that not everything objective is an object. The objecthood of the objective cannot be a correct thesis in his philosophy.

This, I believe, is true also in the realm of sense. Frege, as stated above, was a realist about senses, but never said that senses (or thoughts) were objects, which would be quite remarkable if he really thought so. Moreover, realizing that the superficially tempting objecthood-of-the-objective principle is clearly false pulls the rug under this popular argument for thinking that he did think so. It would clearly be most implausible to suppose that senses of concepts and of functions, which are as objective as anything, were objects. These senses are incomplete like their expressing predicates and their references. Taking them to be objects would make many features of Frege's theory of sense extremely problematic and in fact incoherent. And I believe that the same is true for senses of proper names as well.

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[17] It is worth noticing that the tendency is much less tempting with the German "*Objectivitaet*" and "*Gegenstand*".

It is hard to see, for instance, in what sense a mode of presentation of an object (or for that matter a way of determining it), which is Frege's core idea of sense, is itself a (self subsistent) object. And if it were, the Fregean thesis that sense determines reference would amount to construing the relationship between them as a sort of functional relation, which is clearly **not** what Frege thought.[18] And this may appear even clearer with regard to thoughts than to other senses. Thoughts are objective, but there isn't any reason to think that Frege regarded thoughts to be objects. Facts, for Frege, are senses of true sentences – true thoughts. But facts, typically expressed by propositional that-clauses (that so and so) are not objects. Thinking that they were would amount to there being objects that are true (or false) and objects that are about objects, which are muddled, non-Fregean views. If two contradictory thoughts were two different objects, what is it about them that would make the thoughts contradictory, or that would make one thought imply another? All these seem to be theses foreign to Frege and opposed to his basic principles.[19] Each of the above considerations is sufficient to resist the tendency to ascribe to Frege the view that the objectivity of senses consists in their being objects, and in fact, the view that senses are objects.[20]

There are, admittedly, Fregean doctrines that may **seem** to press in the other direction, notably the doctrine of **indirect reference** in oblique contexts. I discussed all these in Bar-Elli, G.: *The Sense of Reference*, De Gruyter, 1996, particularly ch. 9, trying

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18 SR 64/34-35 in TPF, 158 in FR; PW 194/211; See also his letter to Husserl of 24.05.1891 in FR 149-150.

19 This, as pointed out in Wittgenstein's *Tractatus*, is one of the main problematic features of Frege's view that the two truth-values were objects. But whereas this particular view can be circumscribed, or even discarded, the analogous claim about thoughts and senses in general being objects would make the problem much deeper.

20 I have confined myself here to "purely philosophical" considerations. The view that Fregean thoughts are objects has also raised more logical-technical problems having to do with cardinality and Cantor's theorem, and with Russell-type paradoxes. See e.g. K. Klement: "Does Frege have too many thoughts? A Cantorian problem revisited", *Analysis*, 65.1, 2005 45-49. I shall not go into this, partly because my epistemic construal of sense makes many of these arguments hardly relevant.

to show that they do not imply that senses are objects (in a substantive sense). Frege's theory of indirect reference (mainly in SR) has been held by many to imply that senses are objects, and is perhaps the most serious basis for this view. Let me therefore briefly repeat one of the reasons against it.

Frege held that if a name occurs in an "oblique context" like the complement of "believes" (e.g. the terms after "that" in "John believes that 3+5=8") it doesn't have its regular reference but an indirect reference (*ungerade Bedeutung*), which, in many cases, he suggested, is its regular sense. Thus the indirect references of all the terms after "that" in the above are the senses of "3", "5" etc. Since these terms are names, it then seems to imply that their senses – the indirect references in such a case – are objects.

But this, I surmise, though posing a problem, is not necessarily so. For by all admission these are special cases in which reference is not used in its regular meaning and therefore need not be subject to the principle that the reference of a name is an object (in the full substantive sense). Thus, the theory need not imply that these senses are objects, and Frege's principles can be easily interpreted or emended to the effect that an object is what a name **directly** refers to in **regular**, non-oblique contexts. I also suggested there that the proposition in such a case, though about the senses (the indirect references), it is also indirectly about the regular references, i.e. in our case – the numbers. For, sense is a manner in which reference is given to us. Hence the above is an aspect of the inner relationship between sense and reference and of the supervenience of sense on reference mentioned above. This enables us to explain why John's belief is true and why we can say on its basis e.g. that John believes something true about the number 8.[21] But I shall not further expand on this here.

The view that senses are objects is often supported by what is taken as Frege's **Platonism** about sense. Many scholars speak of Frege's Platonism about thoughts in *Der Gedanke* (T). This is another prevalent view. Beaney (ibid.), for instance, speaks of the Platonism that surfaces in Frege's *Der Gedanke*. But this, I believe, is mistaken. What surfaces in *Der Gedanke* is not Platonism but realism about senses and thoughts. Platonism (in a particular domain) is a specific realistic view (in that domain); not any realistic view is Platonist, and Frege's, in particular, is not. It is easy to drift here into

[21] The claim that Frege's theory of indirect reference does not enable to explain this was raised e.g. by S. Blackburn in (1984).

terminological disputes, but I surmise that part of what distinguishes a Platonist view is not mere realism about concepts (and universals), and not even a belief in their independence of psychological acts, but **the assimilation of them to objects and of their "mode of existence" to that of objects**.[22] Likewise, *mutatis mutandis*, for senses.

This must be also Beaney's view, for he refers to "the Platonism that surfaces in *Der Gedanke*" as evidence for his view that senses, for Frege, are objects. But this mistakenly conflates realism in a certain domain with Platonism in that domain. If the former implied the latter, not only wouldn't Frege be a realist about senses (in *Der Gedanke* and elsewhere), but about concepts as well. For, hardly anyone has insisted on the categorical difference between concepts and objects as Frege did. This is to be distinguished from, and does not conflict with his realism and objectivism about senses, which is the point in *Der Gedanke*.[23] Concepts (and functions in general) are real but are not objects, and so are senses. Frege was a realist about concepts and senses, but (on this meaning of the term) not a Platonist.[24]

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22 See for instance Weiner (1990), ch. 5, though she does not speak there of senses, but only of numbers, which are definitely (abstract) objects, whose knowledge, she argues against Benacerraf, does not require special capacity but just the logical capacity, which is anyway constitutive of thinking and judging. But the point, I believe applies to Platonist approaches to senses as well. Occasionally Weiner insinuates (wrongly I believe) that thoughts are objects, e.g. in saying "The content of Frege's view of thoughts as objective objects is..." (217).

23 I discussed *Der Gedanke* and the alleged Platonism in Bar-Elli (1996) pp. 205-212.

24 My use of "real" "objective" and "Platonist" and the relations between them, as well as my delineation of dependence relations which are constitutive of these notions, differ from e.g. Burge's. Burge emphasizes, along with other characteristics, the absolute unqualified independent existence as a mark of (ontological) Platonism, which he attributes to senses (Burge, T.: "Frege on Knowing the Third Realm", *Mind*, vol. 101, 1992, 633-650, in particular pp. 643-645 and notes 10, 12). I believe this is overstated (as claimed above, there is a sense e.g. in which senses depend on their references), but in any case Burge does not suggest that this makes entities in the third realm, senses included, into objects, which is our main concern.

Senses Are Not Functions (or Concepts)

We can be much briefer about this claim for hardly anyone claimed that in general senses in Frege's theory are functions. There are some proposals by modal logicians to explicate what they take as senses in terms of modal functions using an ontology of possible worlds. This is foreign to Frege, and I shall not discuss it in detail, though some of the following considerations are pertinent to this view as well. Frege did hold that there must be in the realm of sense something analogous to predication and to the distinction between complete and incomplete or unsaturated parts. Otherwise a sequence of senses would not combine to a thought. But this in itself does not say that senses are objects or functions, and the above analogy is only partial. Moreover, the view that senses are functions (on objects), besides its philosophical faults, flatly contradicts explicit pronouncements of Frege's to the effect that **the relation of sense to its reference is not predicative** – it is entirely different from the relation of concepts to objects, or of functions to sequences of objects. But if a sense were a concept (or function) its relationship to its reference would presumably be predicative – just what Frege denies.

Concepts and functions, as said above, are entities in the realm of reference in Frege's theory – they are references of predicates or functional expressions. As anything in that realm they also have senses – when referred to they are given to us in particular ways. These are in some way incomplete. But just as the fact that a sense of a proper name is in some way complete doesn't mean that it is an object, so the incompleteness of a sense of a predicate – a way its reference is give to us – does not mean it is a function. If these ways were themselves functions, they would have to have senses – to be given in particular ways. The result would apparently be an intolerable infinite hierarchy of sense-functions. One might think that Frege's theory of indirect reference is anyhow committed to an infinite hierarchy of senses, even without taking senses to be functions. This was proposed, and logically systematized e.g. by A. Church. But already Russell (in "On Denoting" of 1905) and following him many theorists, notably Dummett, pointed out its philosophical faults. Dummett proposed an amended version that stops the hierarchy at the first level.[25] But regarding senses as functions seems a muddled theory, even independently of the problem of infinite hierarchy: For, in understanding a function we

[25] Dummett (1981a), ch. 9. A somewhat similar idea was in fact proposed by Carnap already in (1956), §30, which Dummett doesn't mention.

must have an independent grasp of its arguments and values. Even granted that at the lowest level their arguments would be regular references (and in higher levels functions of lower levels), what would be their values? Without an independent grasp of these values the alleged functions would be unintelligible.[26] It seems therefore that Frege probably didn't and at least shouldn't regard senses as functions.

On the basis of all these consideration we have to conclude that senses, though real and objective, are neither objects nor functions, but belong to a different sui generis ontological category.

The Category of Sense – Some Characteristics

Some main characteristics of this category were already mentioned above: 1) Senses are real existents, but not objects, nor functions. 2) Senses are objective, which means that they belong to the space of rational reason and justification and do not belong to subjective consciousness. 3) Senses are graspable by the human mind – they are ways in which things in the world, which are the references of our terms are given to us. 4) These ways are constrained by the meanings of the terms concerned, which means that grasping them is internally connected to the understanding of language, and that we conceive things in the world as linguistic creatures. 5) Senses are inter-subjective and communicable. 6) Senses have a justificatory role and belong to the rational space of reasons (either as thoughts or as their constituents). Moreover, it is senses (thoughts), not mere sentences, that stand in logical relations of implication, contradiction etc.

One main point should be added to these characteristics of senses – their **intentionality**: 7) Senses are essentially related to things in the world to which our expressions refer. They are ways in which these references are given. This, as we have seen, is the core idea of sense. Propositions, though they and their constituents

[26] This, with regard to Frege's theory of indirect reference (*ungerade Bedeutung*) in oblique contexts has been pointed out by Dummett ibid. He argues in general that "the sense of a predicate can in no way be regarded as a function mapping senses of proper names on to thoughts" (293).

express senses, are about their references.[27] In grasping a sense, primarily thoughts, our mind is directly related to these things in the world, which are their references.[28]

Following a venerable tradition stemming from Brentano, this property of being about something is called intentionality. It was ascribed by Brentano to mental states, and by many others also to propositions. And I propose that it is characteristic of Frege's conception of senses and thoughts. Notoriously, this view of the intentionality of sense and its intrinsic supervenience on reference seems to clash with a very common view according to which there are senses with no reference so that sense is in this way independent of reference. Though there admittedly is an exegetical problem here, I believe quite the opposite – that this possibility of referenceless sense goes against some of Frege's basic principles. Although some of Frege's formulations (mostly in SR) seem to admit this possibility of referenceless sense, they do not, I believe, express his better and considered view. Rather, this possibility is incoherent with Frege's basic principles, like his adherence to the bivalence principle of classical logic, his view that sense is a way a reference is given to us, and that a meaningful proposition is about the references of its terms.

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27 In SR 58/28 in TPF Frege writes: "**If words are used in the ordinary way, what one intends to speak of is their references** (Bedeutungen)". Frege repeats the principle that the references of our terms is what we talk (and think) about in many places (see for instance his letter to Jordain, PMC 79-80. This is also a principle which is obviously behind Frege's main argument in rejecting the meta-linguistic "solution" to the "identity puzzle" in SR, as well as already in BSsection 8. Dummett justly presents it as a basic thesis of Frege's on reference (*Frege: Philosophy of Language*, 2nd ed. Duckworth, 1981, ch. 6, p. 196 ff.).

28 This is evidently a principle that directs the course of argument in FA – both in its critical and in its constructive parts. Cf. also "17 Key Sentences on Logic", no. 3: "In the case of thinking it is not really ideas that are connected, but things, properties, concepts, relations". In no. 10 he adds: "The sentence 'Leo Sachse is a man' is the expression for a thought only if 'Leo Sachse' designates something (*etwas bezeichnet*)". In his late "Notes for Ludwig Darmstadter" of 1819 Frege writes: "The constituents of a thought do refer (*hinweise*) to the object and concept, but in a special way (*eigentuemlicher Weise*)"

Whenever a term lacks reference, this shows that it only seems to have a sense, but in fact doesn't. Strictly, **sense for Frege is supervenient on reference**.[29] Yet, he conceived of certain **extensions** here in trying to account for our seeming understanding of expressions which only seem to have sense. These extensions relate, according to Frege, mainly to vernacular expressions in mythical, fictional or poetical contexts, which he regarded as devoid of sense in the strict sense. But they may pertain also to scientific and even mathematical contexts (as Frege's own example in SR of "the least rapidly convergent series" shows). In "Logic" of 1897 (in PW) Frege says that in these contexts we have what only appears to be statements (*Scheinbehauptungen*) that contain what only appears to be names (*Scheineigennamen*) and express what only appears to be thoughts (*Scheingedanken*; *schein* was translated in PW "mock").[30] When genuine sentences are used to express genuine thoughts (which are their senses), they cannot contain constituents that are devoid of reference, for then the alleged thought would be neither true nor false, i.e. not genuine thoughts. Expressing genuine thoughts by genuine sentences is therefore an achievement of which in many cases we don't have an a priori guaranty. We may sometimes discover (as we often do) that what appeared to us as a genuine sense (or thought) is not in fact so because it is devoid of reference – it is not a way something is given to us because there is no something to be given. We may be misled in that by linguistic structures and similarities, or even exploit such similarities purposely (as in fiction). But on Frege's view all these are not strict thoughts. It was perhaps G. Evans (1982), ch.1) who first pointed out this and its importance. There followed many objections that claimed that either Evans' interpretation was wrong or that Frege's position is fundamentally incoherent on a basic aspect of it. I believe that the above remarks support

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[29] In contrast to a widespread (but wrong) presentation of senses as "platonic objects". In addition to the above reasons, the naturalness in which Frege could say that senses of indexicals and demonstratives (like "I", "here", "now", "this", etc) contain elements of the outer context (T, in FR 332/64-5) strongly support this interpretation. Many alternatives (including dummett's) are not only exegetically, but also philosophically problematic and make the relationship between sense and "its" reference (when it has one) quite mysterious.

[30] See "logic" in PW 130/141-2.

Evans' position and show that the accusation of incoherence on Frege's part is answerable.[31]

Moreover, my main point is that this intrinsic intentionality of sense is a cornerstone in showing the category of sense to be crucial in explaining our cognitive relation to the world and to things within. The corresponding supervenience of sense on reference, of which we spoke above, marks Frege as a forerunner of the view called "externalism", according to which the meaning of expressions and the contents of mental states essentially depend on the existence of things in the outer world and our cognitive relation to them. In many ways this category of sense cuts across the mind/body distinction, which when sharply presented may pose an insurmountable obstacle on explaining these cognitive relations. Brentano presented intentionality as the characteristic mark of the mental: Unlike physical states, a mental state is essentially about things, which are not its inner constituents.[32] This epoch making observation, when interpreted realistically, so that the intentional state must be directed at real existing objects, besets however a severe initial problem: How can something mental relate to the physical? What is it about it that enables such a relationship? Frege's theory of sense, on its above intentionalistic interpretation, paves a general and fruitful direction for coping with it. An important facet of this is the *sui generis* ontological status of senses, some of whose main lines I tried to sketch above.

Summing up, we can say that in light of a fundamental methodological principle, according to which the basic ontological categories are implied by the logical ones and the requirements of a theory of meaning of linguistic expressions, Frege further revolutionized ontology in at least the following respects: He established a wide notion of reality, including whatever is objective, whether actual or not. Within this wide area he distinguished the realm of reference as consisting of objects and functions - both real and objective but categorically distinct. On top of this he established the realm of senses, which have a sui generis mode of existence and, though real and objective, are neither objects nor functions, are graspable by the human mind but independent of its activities, and are the basic carriers of our knowledge, rationality and our intentional stance to the world.

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31 For further discussion see Bar-Elli (1996), particularly ch. 3.

32 Brentano, F.: *Psychologie von empirischen Stundpunkt* vol. I book II Kapitel I.

Chapter 7: Analyticity, Justification and Objectivity in Frege

The doctrine of analyticity – that there are analytic truths - has been offered as an answer to the problem of apriority: How can there be truths whose knowledge and justification do not rely on (empirical) experience. The relationships between truth, knowledge and justification are thus at the focus, and should direct us in any appreciation of the doctrine concerned. Analyticity is often couched in terms of meaning relations and of truth in virtue of meanings. This is a quite standard conception at least since Carnap. Frege, as we shall see presented a different notion, and we shall later dwell on its advantages over the standard one. Frege's conception of analyticity has been, however, widely misconstrued, and we shall need to take some pains in clarifying it. In particular, the nature of justification, which is central to this conception, and its role in founding the basic truths (axioms) of a domain should be explicated.

Frege's philosophical program in *The Foundations of Arithmetic* (FA)[1] aims at establishing that arithmetic is analytic, in the sense he gives to this term. There is, however, a difficulty, in fact a seeming incoherence, at the basis of the program: The notions of analytic and a priori, Frege states, apply only to what is justifiable. Basic truths cannot be proved. Hence, if proof were the only pertinent form of justification, such truths, including logical ones, could not be said to be analytic. Likewise, a non-logical basic truth, like a geometrical axiom, could not be a priori, and an unprovable singular fact could not be a posteriori. But this seems to be unacceptable. I shall present the difficulty and propose a way out, which, I believe, is important for understanding some other fundamental topics in Frege's philosophy.

The epistemic status of basic truths and axioms raises difficult problems on any view, but particularly in a foundationalist approach such as Frege's.[2] The Fregean direction that I suggest for answering the above difficulty, which admittedly goes at some points beyond what Frege explicitly said, relies on construing Fregean sense (Sinn) as a special cognitive relation to things in the world in a way that makes it fit to

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1 Frege, FA. I emphasize "philosophical" to shunt here aside the formal achievements and the formal difficulties involved in FA, which are not our concern here.

2 The question was already discussed by Aristotle (1941), who claimed that the foundations of a demonstration are not demonstrable – see, *Posterior Analytics*, 72b 10-18. He adds that one cannot ask a reason for everything, (*Met*. 1011a).

belong within the justification-space of what is objective - logical and mathematical basic truths included.

The general philosophical significance of Frege's notion of analyticity, as presented here, lies not only in this epistemic point, but also in suggesting an answer to "the problem of apriority", which is in full harmony with a realistic conception of truth, and with respecting the "principle of the homogeneity" of truth, where its main advantage over the standard conception of analyticity lies.

Proof and Justification

In the central §3 of FA Frege introduces his notions of a priori, a posteriori, analytic and synthetic. These notions, Frege emphasizes, concern not the content of a judgment but its ground and justification (Berechtigung):

"When a proposition is called a posteriori or analytic in my sense, it is a judgment about the ultimate ground upon which rests the justification for holding it to be true".

This, he explains, means that "**where there is no justification, the possibility of drawing the distinctions vanishes**". ("Da, wo diese fehlt, faellt auch die moeglichkeit jener Einteilung weg.") I shall call this "principle J".

Analytic judgments are then introduced, in the same section, the next paragraph, thus: "The problem now becomes that of finding the proof of the proposition, and of following it back to the primitive truths. If, in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one". Evidently, as Frege often explains, in this "backward process" we must eventually come on basic truths or axioms that cannot be proved.[3] It should be noted that this is not strictly a definition (of analytic); it is a conditional that leaves open the possibility of there being an analytic truth that is not proved, and may even be unprovable.[4] Bearing this in mind,

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[3] See for instance "Logic in Mathematics", in Frege, PW, 203-250, particularly 203-5. Frege, though, was well aware already in *Begriffsschrift* that there may be alternative (equivalent) systems such that an axiom in the one is provable in another. See *Begriffsschrift*, e.g. §13). Cf. also PW, p. 205.

[4] In "Frege on Apiority" (Burge, 2005, 356-387) Burge claims that these are definitions stating sufficient and necessary conditions for analyticity and apriority, and gives some reasons for thinking so (359). I remain unconvinced. The reasons given in the text for not thinking these formulations as stating necessary conditions

we shall however, for simplicity, call these conditionals "definitions". Likewise, in the definition of a priori judgment Frege says: "If, on the other hand, its proof can be derived exclusively from general laws, **which themselves neither need nor admit of proof**, then the truth is a priori" (emphasis added). This seems to imply, in light of principle J, that if deductive proof were the only form of justification, the basic truths (the axioms) would be neither analytic nor a priori, which is hard to accept – both in itself and as Frege's view.

What is the relationship between the notions of proof and of justification? Proof, for Frege is deductive inference from truths. He was perhaps the first to formalize and sharply define this notion, where it is presented as a sequence of truths, each being an axiom or derivable from previous truths by means of explicit rules of inference. Proof is evidently a form of justification, but is proof (i.e. deductive inference) the only form of justification? It may seem that justification (Berechtigung), at least in logic and mathematics, just means proof (Beweis), and this is the way Frege's position has been generally understood.[5] I leave here aside what may be called "pragmatical justifications" that pertain to a whole system of axioms – its deductive power, "completeness" etc. – and rather focus on axioms taken individually, in which, for Frege, the primary notion of justification consists. There is an important connection between the two, but its discussion will take us too far afield, and I shall not pursue it here.[6]

– leaving open the possibility of regarding the axioms as analytic - seem to me to overweigh those given by Burge.

5 See e.g. M. Dummett (1991b), pp. 23, 25. Dummett emphasizes that "all justification proceeds by deductive reasoning… he [Frege] does not allow for the possibility of any other form of justification" (25). Cf. also Burge (1998), e.g. p. 315.

6 The general direction of the connection can be appreciated by realizing that there is an internal connection between mode of being given, and being a constituent of thought; these are Frege's two main characterizations of his notion of Sense. A thought, however, is logically constituted by its inferential relations. Hence, modes of presentation, which are the constituents of thoughts, cannot be detached from this logical space of inferential relations. For a fuller account, cf. ***. For a

Coming back to our main topic, the above definitions, as the emphasized part of the definition of a priori makes explicit, presume that in the process of finding or presenting a proof we eventually come on basic truths (or judgments) that "neither need nor admit of proof". Frege evidently did not regard a one-line statement of an axiom as a proof of it; definitely not when proof is regarded as justification. Nor did he admit of hypothetical proofs and reduction ad absurdum, and he would probably object to Genzen-type proof, for he insisted that any proof must be based on true judgments. It thus appears that if proof were the only form of justification (in mathematics and logic), this would mean, in light of principle J, that the above distinctions do not apply to these basic truths or judgments: since they are not provable they would be neither analytic nor synthetic, neither a priori, nor a posteriori. Likewise, particular unprovable facts would be neither synthetic nor a posteriori.

The Notions of Analytic and A priori Should Apply to Axioms

This conclusion, however, seems unacceptable, and contrary to what Frege's position is generally taken to be: Traditionally, logical axioms are the paradigm of analytic truths, and geometrical axioms – the paradigm of a priori ones.[7] This seems also to be the basis of the rationale of Frege's distinctions here: If logical axioms were not analytic, and geometrical ones were not a priori, what would be the point of defining analyticity and apriority as he does? Moreover, Frege was aware, of course, of the possibility of alternative equivalent axiom systems – both in geometry and in logic (BS §13). This means that what is an axiom in one system may be provable (and hence, a priori or analytic) in another. It is implausible however to suppose that the notions of analyticity and a priority are thus relative to a system. This also presses to regarding the axioms as analytic, or a priori. In addition, the justifiability of axioms (including logical ones) may seem to be problematic independently of the applicability of the notions of analytic and a priori to it, and much of what we shall say pertains to it directly. But presuming this applicability our difficulty becomes even more acute. Was Frege then so careless in

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somewhat different approach, couched in terms of the connection between "pragmatical justification" of a system, and the self-evidence of an axiom, see Burge (1998), sections III-IV.

7 The apriority of the axioms of geometry is presumed, for instance, in Dummett (1981b), p. 464.

presenting principle J in this central section of FA? And if not – how should we construe it?

In his FPM (1991b) Dummett says that "with uncharacteristic carelessness, Frege has framed his definition [of analyticity] so as not to cover the initial premises themselves" (p. 24). This is quite surprising. For Dummett, as we have seen, proof is the only relevant form of justification (see note 5 above); hence, he should have regarded principle J, so much emphasized by Frege, as explicitly excluding the applicability of the notions of analytic and a priori to axioms (as well as a posteriori to particular facts). It is not an oversight on Frege's part in formulating the "definitions"; it is principle J that is the source of the problem, and it cannot be regarded as an oversight.

Immediately after the above quoted sentence, Dummett says that Frege's definitions can be obviously "extended" to rate particular facts as a posteriori and axioms as analytic or a priori. He does not formulate the obvious extension explicitly. What would it be like? Dummett does not mean here to emend Frege's notion of proof. He obviously thinks of a slight extension in the "definitions" of "analytic" etc. to rate basic laws of logic as analytic and axioms of geometry as a priori. But unless justification is wider than proof, any such extension would fly flatly in the face of principle J. The axioms cannot be proved. Hence, if the extended definitions would still rate them as, say, a priori (e.g. by stipulating that being an axiom is sufficient for being a priori), this would mean, in light of principle J, that there is a way of justifying them other than by proof. Hence, when principle J is given its due weight, Dummett's correct view that axioms must have been regarded by Frege as analytic or a priori, and particular facts must have been regarded to be a posteriori, implies, in contrast to what Dummett suggests, that the notion of justification concerned in principle J must be wider than that of proof.[8]

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[8] In his (1998) Burge says that Frege "neglects to formulate his notions of analyticity and apriority so as to either include or rule out the foundations of logic" (310; in Burge (2005), p. 322). In note 6 added there he agrees with Dummett that this was an "oversight" on Frege's part, easily emended so as to count the axioms analytic and a priori. In his "postscript to 'Frege on Apriority' " he withdraws this admission, and says that Frege intentionally followed suit with Kant in not regarding the axioms of logic as analytic, the reason being that "such laws are not subject to

Similar comments apply to Boghossian. He is even more sweeping than Dummet and says about all logical truths – not only the axioms – that their apriority (and analyticity) is simply assumed and is unaccountable:

> "So our question concerns only the most elementary laws of sentential or first-order logic. How do we know a priori, for example, that all the instances of the law of non-contradiction are true, or that all the instances of modus ponens are valid?"

As I noted above, Frege thought it obvious that there could be no substantive answer to such questions; he was inclined, therefore, to take appearances at face value and to simply assume the apriority of logic." (Boghossian, 1996)

This, again, is in direct defiance of Frege's explicit view expressed by principle J.

Self-Evidence, Justification and Sense

Some exceptions notwithstanding (e.g. BL p. 127), Frege scarcely talks of axioms in terms of the traditional idiom of self-evidence. He probably heard in this traditional term some psychological overtones he disliked.[9] When he does use it, it is usually in critically discussing others' views (e.g. FA §5). He usually talks of axioms as not admitting, and not in need of proof, and sometimes adds that their truth is independent of other truths (e.g. PW 168). Many scholars, however, do ascribe to Frege the view that axioms, or basic truths as he calls them, are self-evident.[10] But this, even if true, doesn't

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analysis" (Burge, (2005), 388-389). I agree with Burge's recent view that this is not an oversight. Paying attention to the emphasized principle J is sufficient to show this conclusively. Burge, however, concludes that the axioms of logic are not analytic, and axioms of geometry are not a priori. This, as I argue in the text, seems to me unacceptable, and I suggest another way: proof, formal deductive demonstration, is not the only form of justification. The axioms can be justified, but in another way, not by proving them.

9 In his (1998, section IV) Burge, quite convincingly, suggests a distinction between a subjective notion of self-evidence, which he calls "obviousness", and an objective one, which he explicates in terms of an ideal mind. When using self evidence as characteristic of axioms, Frege refers to the latter, not the former.

10 See for instance, Burge (2001), particularly pp. 57, 59, 61; (2005) 360-364; also his earlier (1998), particularly pp. 327 ff. Cf. ***.

seem to solve our difficulty. For first, the relevant notion of self-evidence is far from clear. Frege probably didn't regard the axioms of arithmetic (the so-called Peano axioms) as self-evident in the traditional sense, for his main program was based on the assumption that they are provable, by his definitions, in logic.[11] It is difficult to see in what sense Frege could deny self-evidence to the axioms of arithmetic, and say both of the axioms of (Euclidean) geometry and of his system in *The Basic Laws of Arithmetic* (BL)[12] that they are self-evident. Moreover, Frege was of course aware of the problematic status of e.g. the axiom of parallels in the eyes of many able mathematicians, and of his own qualms about "axioms" other people regarded as self evident, and even about his own axiom V of BL – before the discovery of the contradiction, and after it.

For our concerns here, what is perhaps more pertinent is that it is not clear in what way we can take such self-evidence as a justification. We have seen that on the widely accepted assumption that justification is proof, the basic truths have no justification. The mere dubbing of axioms as self-evident, or even recognizing them as such, is therefore no solution to our difficulty. Being self-evident is never presented by Frege as a proof of a truth, but at most as a reason for its not needing a proof. Therefore, if it is a form of justification, this would just show again that justification is a wider notion than proof. In any case, it would need further clarification that should explicate in what way recognizing a basic truth as self-evident provides a justification for it; indeed, I would propose the following remarks as pointing at the general direction a partial explication of this sort may take.[13]

Frege's repeated idiom, after saying that the basic truths cannot be proved, that they are "not in need of proof", does not resolve our difficulty either. For, assuming that justification is proof, in order for the notions of analytic and a priori to be applicable, the only question, as we saw, is whether and how they can be proved. Whether or not they are in need of proof is a different question.

It seems therefore that the most plausible way out of the difficulty is to realize that Frege held a wider notion of justification – even in logic and mathematics – than proof.

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11 Cf. Burge (1998), p. 336.

12 Cf. also Frege's *Begriffsschrift*.

13 See also Bar-Elli (2016), here chapter 9.

Proof and deductive inference are evidently a basic form of justification. But they do not exhaust it. Axioms and other basic judgments that cannot be proved may yet be justifiable. Realizing this would enable us to understand how Frege could have held that the axioms of logic and of geometry are unprovable, and yet analytic and a priori respectively. I shall outline the nature of this other way in the sequel.

There is, moreover, other textual evidence for believing Frege to have held such a view, which may also hint at the nature of such a justification:

> "Now the grounds which justify the recognition of a truth often reside in other truths which have already been recognized. But if there are any truths recognized by us at all, this cannot be the only form that justification takes. **There must be judgments whose justification rests on something else**, if they stand in need of justification at all. And this is where epistemology comes in" ("Logic", in *Posthumous Writings* (PW), p. 3).[14]

This passage, has been often ignored in the relevant literature, and has hardly been taken seriously. But in light of the above difficulty, in the heart of the philosophical program of FA, it should, I believe, be given its due weight. Frege does not say here what this "something else", on which non-deductive justification may rest, is, besides saying that it is epistemic. This may be an important clue; as I shall suggest, it in fact refers, without using yet the later terminology for that, to his notion of sense, or mode in which something is given to us.[15] I shall add some more evidence for this, with regard to geometry, in the sequel.

Justification and Objectivity

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[14] The editors of the *Nachlass* date this between 1879 and 1891. Since until 1882 Frege's writings were all centered on the *Begriffsschrift* of 1879, I surmise that this piece can be reasonably dated closer to FA, which was published in 1884, or after it.

[15] It should be noted that the general idea of sense is evident not only in parts of FA, but already in BS, §8, where Frege argues that understanding identity statements involve recognizing that names for the same content may differ in their *Bestimmungsweise* – the ways they determine it, which is somewhat akin to his later notion of ways of being given (*Art des Gegebenseins*), or sense.

I shall later return to the nature of this kind of justification. Before doing this, let me mention that recognizing this wider notion of justification is important for understanding fundamental features of Frege's philosophy, other than the status of arithmetic as analytic and a priori – be this important as it is – like his notion of objectivity, and the role of his notion of sense in logic. The criterial sign of objectivity, for Frege, is justification or justifiability: something is objective insofar as it is justifiable or as statements about it are. The first of the three fundamental principles in the Introduction to FA is the demand "**Always to separate sharply the psychological from the logical, the subjective from the objective**" ["Das Psychologische von dem Logischen, das Subjective von dem Objectiven Scharf zu trennen", P.x]. This is one principle, and the two distinctions are parallel. It appears therefore that the objective is the logical. And logic, as Frege makes clear on numerous occasions, has to do with the justification of propositions.[16] The main tenor of FA is to establish the objectivity (or "objective factuality", as Frege sometimes says) of arithmetic by clarifying the grounds or justification of arithmetical propositions. Logic, objectivity and justification thus form an inseparable triad for Frege.

The connection between logic and objectivity, seeing logic as constitutive of objectivity, raises, however, a question, for in its simple sense (and so also in Frege) what is objective is what is "out there" in the world, or what concerns objects in the world; and Frege often further characterizes the objective as what is true independently of recognizing it as such, or independently of the judging subject, and as what is communicable and accessible to different people from different perspectives, etc.[17]

What, then, is the relationship between these two aspects of the objective - the logical aspect, on the one hand, and being concerned with and based on objects in the world, on the other? In trying to answer this, I believe, we must appeal to the third element in the above triad – the internal connection between objectivity and justification. What is objective is only what is justifiable or what is used in a justification; in short, what is in the justification-space. Justification, however, is the business of logic. Logic is thus not only itself objective in this sense, but is constitutive of objectivity by setting up the standards for justification (and hence for objectivity); it

16 See the Preface to BS; FA §3; PW, p. 3; ibid. 147.

17 See for instance FA §26, BL xvii/15, PW 149/137. Cf.***.

is what constitutes the justification-space, and hence – the objective. But being, as it is, inseparable from truth, it also pertains to the world-involving factors – objects and their relations – in a way that is communicable and objective. These objects and relations are always given to us in particular ways, which Frege called *Sinne*, and which figure prominently in justification of claims about them.

Logic and Justification

The main task Frege set up for himself was to establish the objectivity of various domains - particularly arithmetic. For him this means presenting statements in this domain as justified or at least justifiable. Logic was both the paradigm and the primary means of such justification. Deductions and proofs are chains of justifications of some truths on the basis of others – whether within logic or in other domains. In this sense, as noted above, logic is both a paradigm of objectivity, and what sets its standards.

Now, in a narrow sense of logic the standards of justification it sets up are those of proof, or deductive inference: One of Frege's important achievements is that he, perhaps for the first time, explicated in detail the notion of formal proof, and set its standards (this is not to imply that proof, for him, was merely a formal notion; it certainly wasn't). But as noted above, and this is the main point, there is another level of justification, which is not strictly deductive, but can still be considered logical in a wider sense, for logic in this wide sense is the theory of justification or justifiability.[18] This other level is where the basic truths of a domain are justified not by being proved or derived from more fundamental truths (since there aren't any), but in being shown to be clear and evident **by the way "their objects", the things they are about, are given to us**. One can even say that these basic truths are justified by the fact that they express aspects of the ways "their objects" are given us, or, in other words, by the modes of presentation or senses (*Sinne*) of the things they are about.[19]

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[18] See for instance, "logic", in PW, p. 3; "17 Key Sentences…", ibid. 175; cf. also Bar-Elli (2001) (2010).

[19] I talk of "objects" and "things an axiom is about" rather loosely here. Axioms (whether analytic or only a priori) are usually general judgments, and as such are strictly about concepts. Only rarely (as in e.g. axiom V of BL) are objects mentioned in them. However, in a wider sense we can say that the axioms of

Explaining this in detail would require extensive elucidation of Frege's notion of sense, which would take us far beyond the scope of this article. Frege's notion of sense is constituted by ontological, epistemic and linguistic layers, which both constrain and fertilize each other. It is thus at the center of a rich and loaded philosophical junction. Central in this picture is the idea that things in the world are given to us in particular ways, which are, on the one hand, expressed, and on the other, constrained by the senses of the terms referring to those things. They crucially figure, therefore, in justifying basic truths about those things

Frege's primary example of this is geometry. Geometrical truths (theorems) are objective in being justifiable. They are justified, through logical proofs, on the basis of other geometrical truths, and ultimately by the basic truths, or axioms of geometry. But what about these axioms themselves? Obviously, they cannot be proved or derived logically from more basic truths. Should we say they are not justifiable, and therefore not objective, and not a priori? Certainly not: This would ruin the objectivity of the whole edifice built on the basis of these axioms.[20] It is here that this other form of justification gets in: The axioms are justified on the basis of the ways their objects, the objects of geometry, are given to us: "So long as I understand the words 'straight line', 'parallel' and 'intersects' as I do, I cannot but accept the parallels axiom…**Their sense is indissolubly bound up with the axiom of parallels**" ("Logic in Mathematics", PW

arithmetic are about numbers, and those of geometry – about points and lines. This looser way of talk does not affect the main point made here.

[20] Recall the crucial passage from "Logic" (PW 3) cited above: "… But if there are any truths recognized by us at all, this cannot be the only form that justification takes …" Cf. also FA §26. Already in BS (§13) Frege spoke of the axioms (the basic truths) as comprising the "kernel" of the whole system. He emphasize this again with regard to mathematics e.g. in "Logic in Mathematics", PW p. 204-5. With regard to geometry, he insists on the truth of the axioms, and on holding either Euclidean or non-Euclidean geometry to be true – "No man can serve two masters" ("On Euclidean Geometry", PW, p.169).

247).[21] The fact that Frege insisted that geometrical objects and relations are given by ‘‘intuition’’ does not weaken, but rather validates my point. For, it shows that how these objects are given is central to the justification of the axioms. An early recognition of the importance of the way something is given to us to justifying truths about it is present, without yet using his mature notion of sense, already in Begriffsschrift. In §8 there (on Identity) Frege gives a geometrical example of an identity (of two points on a circle), and remarks that the ways these are determined (Bestimmungsweise), which is close to his later ways of being given, is essential to the justification (Rechtfertigung) of the identity.[22] In a piece he probably wrote in the last year of his life Frege wrote: "**From the geometrical source of knowledge flow** (fliessen) **the axioms of geometry**" (PW 273). The axioms then "flow" from something; they have grounds or justification. And this, I suggest, is basically the ways the geometrical things they are about are given – the senses of their terms.

Another example of this kind of reasoning in FA is the following. Frege considers the equivalence: "The direction of *a* is the direction of *b* if and only if *a* is parallel to *b*" ((D(a)=D(b) ≡ a//b). He mentions the possibility of using this for defining parallelism in terms of directions, and **rejects** it on the ground that such a definition does not respect, as it should, the ways things are given to us: "Everything geometrical must be given originally in intuition" (FA, p. 75). Parallelism, he thought, is given by intuition; directions are not. Therefore, defining the former in terms of the latter is "to reverse the true order of things" (ibid.). This principle of respecting the ways things are given to us applies to the axioms as well. The axioms, one may say, express at least some basic aspects of these modes of presentation of the geometrical objects.[23]

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21 I thus think that Burge, for instance, in grounding Frege's belief in the truth of Euclidean geometry just in its longevity, underrates the force of this argument (cf, Burge, (1998), p. 327).

22 This, with a comparison to his later view in ‘‘On Sense and Reference’’ is elaborated in Bar-Elli (2006).

23 Scholars have debated the question of whether this way of being given is basically Kantian spatial intuition, and whether Frege can be regarded as a Kantian in this respect. I shall not discuss this here. Cf. Dummett (1981), pp. 463-470.

Frege sets himself the task of constructing something similar for arithmetic, thus establishing solid foundations for mathematics. The defects he found in arithmetical texts of his day were not only that many arithmetical proofs were unclear or faulty, and that definitions lacked the precision and rigor he required for them, but also that the whole science of arithmetic lacked an objective basis. To remedy the former fault it was necessary, according to Frege, to present arithmetical truths in a systematic logical language that would render their proofs transparent and the basis of it detectable. That was obviously the task of logic, or of constructing a logical language. In principle, this was accomplished by the logical language of his *Begriffsschrift* (1879). But the latter deficiency was no less severe, and called for no lesser a task: the task, namely, of presenting the objects of arithmetic, the objects the axioms are concerned with (namely, numbers), in such a way that the axioms themselves are justifiable. This, in general and programmatic terms, was achieved in (or at least, was the aim of) FA, and more rigorously, and in more detail, in BL.

Frege's logicism - the program of presenting arithmetic as derivable from logic (and appropriate definitions) - was designed to solve both problems. Expressing arithmetical truths in the language of *Begriffsschrift* enabled him to present their proofs completely and systematically. But it served a further and, in a way, more basic aim. Being convinced that there is no other way in which numbers can be construed that can justify the axioms of arithmetic, Frege thought that the only way to achieve this goal was to construe them as logical objects. This, together with a logical rendering of other arithmetical notions, would enable us to present and justify the arithmetical axioms as logical truths. Again, the crucial step here is the double move of regarding the axioms, the basic truths of arithmetic, as being about objects of a certain kind (numbers), and of regarding the ways these objects are given us as justifying these axioms, thus establishing the objectivity of the whole edifice of arithmetic.

This double move may explain Frege's persistent view that establishing the objectivity of arithmetic requires an explication of the nature of numbers and the way they are given to us. The philosophical move towards the definition of the concept of (natural) number in FA starts at the crucial, and extremely condensed first passage of §62, which opens by asking: "How, then, are numbers to be given to us?" And Frege ends (the appendix II to) BL by stating: "The prime problem of arithmetic is the question, In what way are we to conceive logical objects, in particular numbers?" (143).

("Way of conceiving" is one of the expressions Frege uses for his notion of sense, and it is virtually synonymous here with "way of being given".) This was not just a sort of a Socratic wondering about essences; it was rather a requirement implied by the conception of objectivity under discussion, and by the task of establishing the objectivity of arithmetic – the ways its basic truths are justifiable. Frege's logicism is, from this point of view, not only a reductivist program and a technical achievement, but also a philosophical discovery: it is the discovery that one can present the basic truths of arithmetic as being about objects, whose modes of presentation and the ways they are given to us as logical objects justify these truths.

This is true of logic as well. The basic truths of logic (e.g. of BL) are not provable, and yet claiming them to be analytic means that they are justifiable. This was established in previous sections. What I have proposed here is that they are justifiable by explicating the modes of presentation of "their objects" – the things they are about. This, I have suggested elsewhere[24], was Frege's main reason for recognizing "logical objects", like extensions of concepts and truth-values, and for incorporating *Sinne* – modes of presentation – into logic, conceived in the wide sense as the theory of justification. Truth-values are given to us, Frege thought, in ways that are (partially) expressed by the basic truths of propositional logic; and extensions – by those of predicate logic (including some form of axiom V of BL). The idea is that the basic truths of logic and arithmetic are secured their objectivity (and their being analytic) by being justifiable – not by proving them (which is impossible), but by presenting them as expressing features of the *Sinne* – the ways their objects are given to us. And this is also why Frege was so troubled by the inconsistency pertaining to his axiom (basic truth) V of BL: It was not just a logician's worry about inconsistency; it presented also a deep philosophical problem. For, without this axiom, he saw no way in which classes – extensions of concepts and functions – are given to us, and no way in which numbers are given to us; and without this - no justification for the axioms governing them, and no basis for the objectivity of arithmetic.

Analytic in the Narrow Sense and Analytic in the Wide Sense

We can come back now to the definitions of analytic and a priori with which we began, and to the status of principle J. We should first recall that these are not strictly

[24] Bar-Elli (2001).

definitions, but (one-sided) conditionals, which leave open the possibility of there being analytic (and a priori) truths that are not proved, and are even unprovable. Secondly, we should realize, that, in light of our discussion above, principle J is concerned with justification or justifiability – not necessarily proof or provability. In the narrow sense, justification in mathematics and logic is basically proof: a truth, accordingly, is analytic in the narrow sense, if it has a proof, which relies only on general logical laws and definitions. It is a priori, in the narrow sense, if it has a proof, which relies only on general laws (not necessarily logical). What, then, about the basic laws or the axioms themselves? In light of our discussion above we can say that although they are evidently unprovable (within the system concerned), this does not necessarily mean that they are unjustifiable. If they were, this, in light of principle J, would mean that the distinctions concerned (analytic / synthetic, a priori / a posteriori) would not apply to them, and we could not say of logical axioms that they are analytic, or of geometrical ones that they are a priori.

But this hard-to-accept-conclusion does not necessarily follow. For, these axioms are justifiable: Indeed, they are unprovable, but they are yet justifiable in being self-evident by expressing aspects of the modes in which the objects they are about are given to us. This is a wider sense of analytic and a priori in which we can say of the axioms of logic that they are analytic, and of the axioms of geometry that they are (synthetic) a priori, in agreement with Principle J. Frege does not explicitly introduce these wider notions, but there is nothing in what he says that prevents introducing them, especially not principle J, and our discussion above shows that the general meaning of what he says gives much support to introducing them, provided we recognize that there is a wider notion of justification than deductive proof. We could then introduce these wider notions of analytic and a priori thus: A truth is analytic if (a) it is provable from general logical laws and definitions, or (b) it is not in need of such a proof, because it is justified by dint of being self evident in expressing features of the modes in which the things it is about are given to us. Analogously (without the restriction to logical laws), a truth is a priori if (a) it is provable from general laws, or (b) it is a general law, which is justified in being self evident by expressing features of the modes in which things it is about are given us. The definitions concerned in (a) raise grave difficulties. They are usually not just unconstrained stipulative definitions (in which case any consistent first order theory would become analytic), but are what Frege calls ‘‘analytical definitions’’,

which must satisfy severe philosophical constraints. I cannot expand on this here (see notes 27 and 30). Analogously (without the restriction to logical laws), a truth is a priori (in the wide sense) if (a) it is provable from general laws, or (b) it is a general law, which is justified in being self evident by expressing features of the modes in which things it is about are given us.

In thus speaking of the things a truth is about, the ‘‘things’’ concerned may be either objects (including extensions or classes) or functions (including concepts). It must also be remembered that Frege held a quite special view about what general truths are about – namely, that they are about functions (concepts). The universality of logic, for Frege, does not consist, as some people think, in its being about "everything"– truth-values, concepts, extensions, chairs, flowers and whales. No. A logical law is not about any chair, or any flower or any whale. This is clear concerning the laws of propositional logic, which, in the mature Frege, are about truth-values. But it is also true about those of predicate logic. Unlike many post-Tarskian views, a (universally) quantified law is, for Frege, not about the values of its bound variables, but about concepts, or, after introducing axiom V of BL, about extensions of concepts.[25]

I have also spoken about axioms as expressing ‘‘features’’ of the ways the things they are about are given to us. There is a hierarchy here. Speaking of concepts, logical axioms express the most general and abstract features of the ways any concept is given to us—the concepts of a straight line, flower and whale included. Each of these, however, has special other features of ways it is being given, which are the business of other branches of knowledge to express. And where the branch is axiomatic, like Euclidean geometry, they may be expressed by its axioms. The special status of logic in this hierarchy is the root of the distinction between analytic and other a priori truths.

At the beginning we have remarked that in light of principle J, it might also seem that there are singular factual truths, which are neither synthetic nor a posteriori. This is also hard to accept. The wider notion of justification we have tried to explain can open a way of accounting for this as well. For, it seems that some factual (e.g. percpetual) truths can also be justified by expressing aspects of the modes in which their objects are given to us. "This tree is green" is justified, in appropriate conditions, by looking at the tree. This may be regarded as justification because of the ways in which colors (and

[25] FA §§46–47; cf. Bar-Elli (1996), Chap. 7, especially 183–195, and Bar-Elli (2001).

trees) are perceptually given to us.[26] Again, discussing this in detail, however, would take us too far from the ostensive topic of this paper.

Analyticity: The Kantian Heritage, Frege and Carnap

The notion of analytic has a long history into which we shall not pledge here. But in order to clarify my point about Frege's notion of analytic and the special form of justification the analyticity of axioms evoke, it would be profitable to briefly compare it with two other landmarks in its history - Kant, on the one hand, and Carnap on the other. For Kant, a judgment is analytic when the concept of its subject includes the concept of its predicate. Kant connects this, in a way which is not entirely clear with the "law of contradiction". In a famous passage of his *Critique of Pure Reason* (Kant 1933) he writes:

> "In all judgments in which the relation of a subject to the predicate is thought […] Either the predicate B belongs to the subject A, as something which is (convertly) contained in this concept A; or B lies outside the concept A, although it does indeed stand in connection with it. In the one case I entitle the judgment analytic, in the other synthetic. Analytic judgments (affirmative) are therefore those in which the connection of the predicate with the subject is thought through identity" (B10, p.48).

Kant then speaks of synthetic judgments as "expanding" our knowledge, while analytic ones are merely explicative, analyzing the subject-concept into what is already thought in it. A judgment is a priori, for Kant, if its knowledge is independent of experience (ibid. B2-3). Kant's formulation intermingles, quite confusingly, logical, epistemological and even psychological terms. But it suggests, as generally conceded, that the analyticity of a judgment concerns its content: the inclusion of one concept in another; its apriority – the way its truth can be known.

Frege probably thought otherwise. In §3 of FA he emphasizes that according to him the distinction between the analytic and the synthetic, like the one between a priori and a posteriori, concerns not the content of a judgment but its possible justification. To this he attaches a note saying that by this he means to state clearly what other writers, notably Kant, have meant. Whether Frege was right about Kant or not, it is clear that on

[26] Cf. Frege's discussion of ''snow is white'' in FA, §26. For an elaborated discussion, cf. Bar-Elli (1996), 35–43.

his own view the notion of analytic pertains to the justification, not the content, of a judgment. It is worth mentioning here that Frege's footnote does not concern his "definitions" of analytic etc. as many have thought (cf. Dummett FPM (1991), chapter 3), but only his saying that they pertain to the justification of a judgment. Hence, this note is no basis for thinking that Frege underrated or was not fully aware of the novelty of his notions of analytic, etc. and of the difference between them and those of Kant.

As many have observed Kant's notions, in being restricted to subject-predicate judgments, are of a rather limited applicability. His notion of concept and the meaning of "containing" in his formulation are not clear. Evidently, Frege could not be satisfied with these formulations. In any case, it was important for Frege to emphasize that his notions concern the possible justification of a judgment, and it is in this connection that he formulates principle J.

Moreover, in the conclusion to FA Frege clearly states that given Kamt's notions of analytic and synthetic, the conclusions he [Frege] draws from his definitions ought to be regarded as synthetic! (FA 101). This statement, so often ignored in the literature, clearly suggests that he was not only sharply aware of the novelty of his notion of analyticity and of his re-construing the analytic synthetic distinction, but that he saw it as crucial for appreciating the difference between his position and Kant's on arithmetic.

Carnap, on the other hand, introduced a notion of analyticity, which is clearly concerned with propositions (not judgments) and meaning relations between the words comprising them. Hence, this notion pertains to the contents of propositions, not necessarily to their justification. Following him this has become the prevalent notion of analyticity, both by adherents and opponents. A proposition is analytic, on this conception, if it is true in virtue of the meaning of its component words. By "meaning" here is meant conventional meaning of a word in a language. Within his formal semantic systems Carnap explicated this notion in terms of his distinction between the extension and the intension of a term: Analyticity and the meaning relations constituting it are a matter of intensions. While the truth (or falsity) of a regular non-analytic proposition, like "Socrates is wise", is a matter of relations of the extensions of its terms, namely Socrates and the extension of "wise", the truth of an analytic one, like "A square is a rectangle" depends only on the intensions of these terms, which is a linguistic matter. In order to know its truth one needs not examine facts (extensions-

relations) but simply understand the proposition – know the conventional meaning of "square" and "rectangle".

Obviously, when the terms concerned are not defined by stipulation, but are meaningfully used, determining their intensions and the analyticity of propositions containing them, may require an analysis. But this, on Carnap's conception, is an analysis of linguistic meanings, not of matters of fact (except facts of linguistic usage). In a formal semantic system such an analysis can sometimes be expressed by a set of "meaning postulates", on the basis of which the analyticity of corresponding propositions may be proved. His position on this raised a vehement critique and opposition, notably by Quine, who maintained that this kind of distinguishing matters of facts and matters of linguistic meanings cannot be generally maintained.

I mention this quite well known story, not in order to enter a discussion of it, but merely as a background for a sharper appreciation of Frege's notion of analytic and the point I have made concerning it. The truth of axioms, and their being self-evident are rooted, I have proposed, in the senses (*Sinne*) of their components, i.e. in the ways their references, the things the axioms are about, are given to us. This could lead us to think that his position is quite close to that of Carnap (as Carnap himself, who studied with Frege, thought). But I think that this is a mistake: Meaning relations, for Carnap, are between the conventional meanings of words; they are entirely linguistic or at the "linguistic level". Things in the world and ways they are given to us are not in the game. Fregean senses, on the other hand, are epistemic; they consist of our cognitive relations to things in the world; they are ways in which things in the world are conceived by us. If we imagine language and the world as two planes in space, we could say metaphorically, that Carnapean meaning relations are confined to one plane – the linguistic. Fregean modes of being given, in contrast, have a vertical dimension; they relate both planes with one another. They are modes in which things in the one are conceived and expressed in the other. This is the deep reason why it was so important for Frege to emphasize that his notion of analytic is concerned with epistemic justification – the ultimate grounds of truths and of justifying claims to knowing them.[27]

[27] This may have escaped Boghossian. He distinguishes an epistemic notion of analyticity from a metaphysical one, and says of the former: "According to Frege, a statement's analyticity (in my epistemological sense) is to be explained by the fact

The Significance of Frege's Notion of Analyticity – Truth

I have been arguing that in contrast to the standard interpretation, Frege's notion of analyticity is based on justification, and that justification is not solely deductive or inferential: there also is a sort of "epistemic" justification, in which a statement is justified by dint of the senses (*Sinne*) - modes of being given - of the things it is about. In this way one can understand the analyticity of the axioms of logic, the apriority of the axioms of geometry, and the aposteriority of singular factual statements, all of which are unintelligible, if the only form of justification is deductive inference.

Getting clear about Frege's exact view is important enough. But one could still wonder what philosophical importance this may have beyond the details of Frege's exegesis. Suppose I am right about Frege's position; is there, beyond exegetical meticulousness about his position, a special significance or advantage to his position over the standard one? I believe the answer is positive and shall try to explain some additional aspects of its significance.

A general background question here pertains to the significance of the notion analyticity in general – not particularly that of Frege's. Now, analyticity has been regarded a major philosophical topic (even if not under this name) at least since Leibniz and Hume. It was regarded by Quine as one of the two important "dogmas of empiricism", and many regard it as at the centre of the main business of philosophy – conceptual analysis. It may still not be clear why it was regarded to be of such a philosophical significance, and in what the significance of the general problem of analyticity consists. With regards to this general question I shall generally say that the significance of the notion of analyticity is in the challenge it poses to our conception of truth and the relationship between truth and meaning. For many realists (and empiricists) truth consists in some "correspondence" with the world. For many idealists (and rationalists) it consists in an inner notion of coherence within our thoughts, ideas

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that it is *transformable into a logical truth by the substitution of synonyms for synonyms*. When a statement satisfies this semantical condition…". As stated above the relevant kind of definitions ("what Frege calls analytical definitions" are not statements of synonimity or sameness of conventional meaning, but as explications of modes of being given. Therefore, "semantical" in Boghossian also seems inappropriate.

and beliefs. This is a large and metaphysically loaded cleavage. However, either schools (or best proponents of them) seem to subscribe by a principle we may call the "homogeneity of truth": Whatever your metaphysical view of truth is, it should not recognize different kinds of truth but should be homogenous across the board (of propositions or thoughts).

Now for idealists and rationalists the significance of analyticity is obvious: it can be, and often has been, regarded as the paradigmatic case of truth in general. Leibniz, for instance, thought that any truth is basically analytic – of the form (he called identity) AB=A (what is A and B is A). Truth, essentially, is the inclusion of the predicate concept in the subject concept. Contingent and singular truths seem in this respect problematic, and Leibniz devoted much effort to explain their conformity to his general conception of truth, in terms of the distinction between finite and infinite analysis of concepts, and the claim that their "analytic" character is revealed only by infinite analysis, which is unavailable to us, and which only God can see.

The important point for our concerns is not only the centrality of analyticity in Leibniz's conception of truth, but also that Leibniz's theory seems to be motivated by an insistence on a homogenous notion of truth – at bottom we have one homogenous notion of truth – the inclusion of the predicate-concept in the subject-concept - which is in principle analytic. His metaphysical effort was to explain, on this basis, two things: contingency (or what appears to be contingent), and the realistic character of his analytic notion of truth.[28]

For many realists (and realistically inclined empiricists), truth consists in some kind of correspondence to facts in the world. There are many variants and characterizations of the realistic position (Aristotle's, Frege's, Wittgenstein's (the Tractatus), and Dummett's are notable examples). But on all variants the gist of a realistic position is that the world and things in it are the essential and decisive factors in determining the truth or falsity of our sentences and thoughts: "In order to tell whether a picture [proposition, thought] is true or false we must compare it with reality" (2.223).[29]

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[28] This has been argued in detail in Bar-Elli (1985).

[29] Wittgenstein: *Tractatus Logico Philosophicus* (1961). Reality, in the Tractatus is the totality of facts, or facts in logical space, and facts are "existing states of affairs". All these raise notorious questions we shall not deal with.

"It is impossible to tell from the picture alone whether it is true or false" (2.224). And the world, in this determination, is "autonomous", and is not subject to or constrained by anything. This simple and basic idea is crucial for a realistic conception of truth. I shall call this in short "the autonomy of the world".

Now, analyticity poses a problem for this conception of truth, for it seems that in the case of analytic propositions the world is not playing any role at all – these propositions are true, according to the standard conception of analyticity, by dint of the meanings of their terms alone, or in Quine's phrase – an analytic truth is "grounded in meanings independently of matters of fact" (TD). Quine had here mainly Carnap in mind, but he almost repeats this formulation when summarizing even Kant's view: "Kant intent can be restated thus: a statement is analytic when it is true by virtue of meanings and independently of facts" (ibid).

But what then is the meaning of "true" here? And what is the connection between such truths and the regular realistic conception of truth as described above? The notions of meaning and of relations between meanings are problematic and have been attacked by many philosophers. But even disregarding this, and supposing that we have some satisfactory notion of meaning and of meaning-relations, what then do such meaning relations have to do with the autonomy of the world in determining the truth of propositions, as stated above?

This then is the challenge analyticity poses to our conception of truth. "Challenge to our conception of truth" should be underscored. Quine, as is well known, opposed the very intelligibility of the standard notion of analyticity. The gist of his opposition was based on his doubts concerning the intelligibility of the notions of meaning or necessity, on which this conception allegedly relies. He devotes much space to showing that one cannot define analyticity in terms of these notions in a non-circular way. This opened a live and wide discussion of these matters for at least a generation. Our questions here, let us emphasize, are different. Suppose there is a satisfactory retort to Quine's strictures, and we have an intelligible notion of meaning, on the basis of which we can speak of meaning-relations, and can talk of a certain property of propositions – analyticity. Why should we think that this property is related to truth? Why should we think that propositions having it should be true? And if for some reason we insist on talking this way, in what sense are they true, and what would be the connection between this "analytic truth" and the realistic notion of truth we were talking about before? For it

seems that the role of the world, its objective autonomy, which were the core of the notion of truth, faint and disappear here. The world and its various situations do not play here a role any longer, and suddenly, in their stead, we have meanings and truth by virtue of meanings.

On this background we can better understand, I believe, the main significance of Frege's notion of analyticity. In contrast to the sort of duality in the conception of truth implied by the standard conception of analyticity, the Fregean conception succeeds in maintaining the homogeneity of the concept of truth. There are no different kinds of truth – one "by virtue of facts" in the world and one "by virtue of meaning" and meaning-relations. There is rather one notion of truth and the distinction between the analytic, synthetic, a-priori and a posteriori concern kinds of justification and bases of knowing these truths. There is no appeal in this explanation to a notion of "truth by virtue of meaning", but only to various justification-procedures, which we need recognize in any case.

One could object here that "by virtue of meaning", in the standard conception, is also a procedure of justification, and that therefore the difference between the standard conception and the Fregean one is not as substantial as I represent it. But this, I believe would be wrong, and the difference in question remain valid: In the standard conception the world gets out of the game (of justification) and does not function at all in determining the truth or falsehood of analytic propositions. Not so in the Fregean conception. A truth, on this conception, is analytic if it has a justification relying only on logic and definitions. Both logic and definitions (in the relevant sense) concern, on Frege's conception, the world: Logic, for Frege, is a universally true theory (of the world); and the definitions concerned in such a justification are not stipulative, or conventional determinations of meaning, but rather explications of the ways in which the defined (or what is denoted by it) is given to us.[30] These ways of being given

[30] This has been argued in detail in Bar-Elli (2009) and (2016), where I argued that the course of argument of Frege's FA expounds a sophisticated view of the nature of philosophical analysis, in which analytical definitions (to be distinguished from "constructive" definitions) are part of a philosophical explication of ways in which the things concerned are given to us, which respects some fundamental principles of Frege's philosophy

(Sinne) are rooted in the things in the world and their properties, and therefore the notion of truth that is founded on them leans, just like in regular truths according to the realistic conceptions, on the world, and the things in it. Hence, the notion of truth involved in analyticity, on Frege's conception, is not, in principle, different from the "regular" notion, which respects the autonomy of the world, and according to which things in the world determine what is true. The problem of apriority concerns justification and ways of knowing. Analyticity, which is a (partial) answer to this problem, is likewise construed.

The above point can be summed up as follows: The significance of the notion of analyticity lies in the kind of answer it suggests for the problem of apriority – how can there be truths whose knowledge does not depend on experience. The problem of aprioirity may be stated thus:

1) Truth consists in correspondence to facts.

2) Facts are known only by experience (the empiristic assumption).

These two seem to exclude what was taken to be an obvious truth:

3) There are many a priori truths – truths known independently of experience.

These formulations are very rough, and they are unfaithful to many of the philosophers concerned (including Frege). They are intended here just for a general picture I wish to draw, and can be adjusted in each case more carefully. In these rough terms Carnap and the classical tradition answered the problem by denying (1): there are also truths "by virtue of meanings" (analytic truths). Their being true consists in meaning relations; not in correspondence to facts (Carnap 1956). Frege rejects this and answers the problem by denying (2): There are also objective facts, which are not necessarily known by experience, but by logic and reason alone. In this, Frege partly followed Kant who also denied (2) by proclaiming intuition (Anschauung) in addition to sensual experience as a source of knowledge. Frege had special reasons against applying this to arithmetic and logic. I shall not go into them here.

It is worth noticing that there is from this perspective an important difference between **analyticity and necessity**. In a necessary proposition (like a logical or mathematical one) it also seems that the world cannot change much – there are no options and degrees of freedom there – and the world and the situations within it don’t

seem to play a role in determining the truth of the necessary proposition. But, in the sense in which this is true, the claim here has a different meaning from the one it has with regard to analytic propositions. In a necessary proposition the world is getting closed, so to speak, from within: the space of possibilities is there, and is functioning, but the proposition turns out to be true in all of them, so that the differences between them don't count. This is an edge point or a degenerated case of the classical realistic conception. The world does function here and determines the truth of the necessary proposition; the determination is degenerated though, in the sense that it is like a constant function – it is the same in all situations. In this sense, necessary propositions do not threat the homogeneity of truth – the world, is an argument in determining the truth of a necessary proposition, and the latter is a function of the world, though a constant one.

Analytic truth, in the standard conception, is an altogether different story. The space of possibilities is not getting closed here "from within", so that it turns out that the proposition is true in all of them, but "from without", and a priori. The space really doesn't function here – not that it is found out, at the end, to have a constant result. Therefore, "truth in virtue of meaning" is an altogether different game from "truth in every possible world", even though they may have extensionally the same results, and even though the results in both of them are similar, in the sense that the actual situations in the world don't make a difference. In the case of necessity, the proposition is true in all possible worlds – that is in $\alpha 1$, $\alpha 2$, $\alpha 3$, and so on in all the αs. The case of analytic truth is entirely different: there is no "true in α" there at all, and the different situations α are not in the game here from the very beginning. This is part of the reason why seeing a necessity often requires proof, and sometimes very hard and complicated one. It is not so with analyticity (unless one stipulates necessary truths to be analytic).[31]

Frege's notion of analyticity renders it much closer to necessity than in the standard conception; and both of them – to the classical realistic notion of truth. They

[31] This distinction between the analytic and the necessary seems to be generally undermined or missed altogether. Even Kripke (1972), who sharply distinguishes the necessary from the a priori construes necessary propositions as analytic. This, in the case of de re metaphysical necessities, which are his main concern, is even less persuasive than in the standard cases.

are both in the same kind of game – the justification game – in which things in the world and the modes they are given to us have crucial role in any case, even when at last it turns out that the propositions concerned cannot but be true. Frege's conception of analyticity is couched in full harmony with the principle of the homogeneity of the notion of truth mentioned above, and does not require or suggest the split between truth by facts and truth by meaning, characteristic of the standard conception. In this, Frege and Quine are proceeding in the line of Leibniz in keeping up the homogeneity principle; the standard conception, Kant, Carnap and the notion Quine set up to attack do not.

The Significance of Frege's Notion of Analyticity - Knowledge

We talked above of Fregean sense – a way something is given to us – as providing a form of epistemic justification for propositions about it. But what is the justification for our talk of "epistemic justification" here? Justification, so it seems, consists of rational, logical relations, like implication relations, between propositional contents. How can a way in which something in the world is given to us, which apparently is a non-propositional "objectual relation", be part of it? How can it belong to this "space of justifications"? This, in one version or another, is of course a cardinal problem of epistemology, and many philosophers have tried to pave their own ways in trying to solve it. Frege unfortunately did not address it directly, and I shall not discuss it either. It is too big a topic for the scope of this paper. I wish however to point out that the Fregean picture I have been trying to portray here seems an important (and rather neglected) approach to addressing it. On this picture of sense (Sinn), it is a three-folded notion: sense is a mode in which things in the world are given to us; sense is a component (a "building block") of thought; and sense is the meaning of an expression in a language.

Some scholars regard this multiplicity as confused and misleading. This accusation would be justified if the three dimensions concerned were not internally interrelated and just stipulated as belonging to sense in an arbitrary way. The Fregean theory of sense and reference, however, displays such intrinsic and internal connections between these three, that they may be seen as three inter-related facets, mutually constraining each other, of the same notion. As explained above, the first facet is mainly what tells Frege's notion (of analyticity and meaning) apart from Carnap's and the standard conception. But it is the second facet, constrained by the objective criteria

embedded in the third, which provides the notion of sense with its rational and propositional dimensions, by dint of which it can justly be seen as "epistemic" and as belonging to the space of justification and knowledge. This feature of sense may be valid and relevant to any kind of knowledge and justification, but it is especially conspicuous with regard to the justification of basic truths and axioms - hence, its central status in understanding Frege's notions of analyticity and a priority.

Frege, as mentioned before, said so little on this subject that it is difficult to ascribe to him any clear epistemological position here. But it seems that the general tenor of what he says, as portrayed above, makes it appropriate to ascribe him with a sort of an intermediate position between two alternatives. On the one hand, if one sees mode of being given as a justificatory basis of knowledge, it seems natural to impute propositional content to it. Thus, in regular perception, as when I see a red flower in front of me, an object may be given to us in a particular way, which may justify my claim to know that it is thus and so – that the flower in front of me is red.[32] On the other hand, it seems that the whole point of appealing to seeing the flower and its color is that it is a naturalistic, "objectual" relation, which forms a basis for this propositional content and the claim to know it. From this point of view it cannot carry this content itself, for then its role as a base or ground for the claim would be circular and vacuous. It is difficult to see, to put it in other words, how such a naturalistic relation to an object can serve as a rational base or reason or justification for a propositional content and the claim to know it. In itself this naturalistic objectual relation just doesn't seem to belong with the rational and logical relations between propositional contents – to the space of justifications.

The intermediate position I have alluded to, which emerges from the Fregean three-fold notion of sense, tries to find a hold on both horns of the dilemma: according to it, a sense - a mode of being given - is indeed of things in the world (and is a kind of objectual relation – the first facet). It is, however, rational, and not merely naturalistic (the second facet). Seeing the red flower does not in itself have a propositional content in the full sense of the word – it is different from seeing that the flower is red. But it enfolds a particular way of conceiving of the flower and its color, a particular way in

[32] In FA this kind of examples is given by Frege as the paradigm of what is objective, where the objective, as argued above is what belongs to the justification space.

which it is given to us, which, being as it is a component of thought, is pregnant with rational dimensions – a potential for justifying a propositional content. It is impossible to say in general what this mode of being given is; it obviously changes from case to case, but it is important to realize two things here: First, it is expressed precisely in its being seen by us as a potential basis for a propositional claim, as capable of justifying a propositional content. And second, it is objectively constrained by the meanings of the relevant expressions and the relations between them imbued in language. This highlights one of the main aspects of the significance of the triple-foldedness of Frege's notion of sense, namely, that we conceive of the world and things in it as linguistic creatures.

Chapter 8: Three Kantian Strands in Frege's View of Arithmetic

One problem confronting the idea that arithmetic contains genuine knowledge is its analyticity: if arithmetic is, as Frege famously proclaimed, analytic,[1] doesn't this mean – as Kant and many of the positivists thought – that it does not express genuine objective knowledge?[2] Frege's answer, as we shall see, is No. His position here is quite remarkable in holding both that arithmetic is analytic and that it contains genuine objective knowledge and "extends our knowledge". In the course of elaborating this I shall indicate the difference between Kant's and Frege's notions of analyticity, yet try to elucidate three Kantian strands[3] in Frege's view:

1. Arithmetic, though analytic, extends our knowledge, and in Kant's sense of the term is synthetic.
2. Arithmetical (and logical) knowledge depends on our capacity to recognize objects.
3. The reduction of arithmetic to logic (via appropriate definitions) displays a version of Kant's leading idea of analyticity – the idea of conceptual containment.

On the relationships between Kant's and Frege's views on mathematics, there is a standard story going somewhat like this: Frege was a Kantian on Geometry, accepting

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1 I focus here on *Die Grundlagen der Arithmetik* (Breslau, 1884) – *The Foundations of Arithmetic* (FA). One should bear in mind that Frege was clear then that he still hadn't shown this conclusively, and was aware of possible doubts (FA, §90).

2 This is a widespread, almost standard, conception: For a recent instance, see S. Yablo (2008) where he characterizes Frege's notion of analyticity as uninformativeness (p. 154). This in Yablo is a casual aside, which doesn't bear on the important content of his paper, but just for that it reflects a widespread conception.

3 "Kantian strands", because some of the pertinent Kantian ideas are unclear, and though as I shall argue, they play a role in Frege's view, this does not mean that he endorsed them as they are. These three strands go beyond Frege's general applause to Kant's raising the analytic/synthetic distinction with respect to mathematics – reflecting, as it does, a search for the "ultimate sources of our knowledge" – and for his doctrine of the synthetic a priori (FA, §89).

Kant's view that geometry is synthetic a priori, but rejected Kant's view that arithmetic is synthetic, and argued, or even proved, that arithmetic is analytic.[4]

There are some misleading points in this formulation. Frege's view of arithmetic would amount to rejecting Kant's if they were using "analytic" and "synthetic" in the same way. But they were *not*. First, though the point has been debated in Kant's scholarship, most commentators concede that for Kant, analyticity is a property of the *content* of a proposition (or a judgment), namely, that its subject (-concept) contains its predicate (-concept)[5]; for Frege, it concerns its *justifiability*, namely, whether it can be justified on the basis of logic and definitions alone (more on this in the sequel). Though there is obviously a connection between the two, there are also important differences. And secondly, Kant's notion, besides being too restrictive in applying only to subject-predicate propositions, is notoriously hazy and unclear, as the relevant notions of subject, predicate and containment are, in contrast to Frege's wider and much more precise one. In fact, at least in Frege's eyes, the differences are so significant that, as we shall see in the sequel, Frege explicitly says in *The Foundations of Arithmetic* (FA) that by Kant's notions of analytic and synthetic, arithmetic should be deemed synthetic!

Likewise, whether or not arithmetic is justifiable on the basis of logic and definitions depends on the nature and scope of logic and on those of the acceptable definitions**.** Evidently, Frege's and Kant's ideas on these differ substantially. Many scholars argued that the differences in their views of logic are so vast and basic that a serious debate on the nature of arithmetic and its reducibility to logic is unintelligible. Others contested this view and argued that the great differences in their views on logic notwithstanding, there still is a shared basic core of their conceptions of the essence of

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4 I shall leave geometry aside here. Let me just mention that Frege's admiration of Kant inclined him to undermine their differences (cf. FA, § 89 second paragraph). This bears also on their use of "intuition" (*Anschauung*), which Frege, unlike Kant, applied also to concepts and functions. I shall also ignore his possible change of view after the "Russell paradox", and particularly at the last year of his life.

5 See the Introduction to the *Critique of Pure Reason* (CPR), B11. There are several other formulations in Kant, in terms of the "law of identity" and the "law of contradiction". Most commentators agree that they all concern the content of a judgment, though some have challenged it; see, for instance, van Cleve (1999).

logic, which makes a serious debate here possible – both on the scope of logic and on the nature of arithmetic and the logicistic thesis.[6]

I shall leave aside here the nature of logic and the important differences in Kant's and Frege's views of logic and its scope, and claim that even assuming that Kant would, or should have accepted Frege's logic, there is much amiss in the standard story, and the above difference in their notions of analyticity (apriority etc.) is decisive. But yet, looked more deeply, **Frege's conception of arithmetic has important Kantian strands**,[7] where a key to understanding this is to note some aspects of his notion of sense, which have been rather ignored or played down in the literature. I should emphasize that my main concern in this paper is with Frege's view, not Kant's. I shall therefore allow myself to remain quite general and imprecise about Kant. This, I hope, should not be too detrimental to my detecting general Kantian strands in Frege's view. On this background it may be interesting to examine various interpretations of these strands in Kant and their effect on the theses proposed here, but this would go beyond the scope of this paper.

Sense and the Justification of Axioms

The notion of sense is central in Frege's philosophy, even in his writings before he introduced his systematic terminological distinction between sense (*Sinn*) and reference (*Bedeutung*) in the early nineties. Sense is primarily "mode of being given" or "mode of presentation" - *Art des Gegebenseins* – in which something (an object, or function, including concepts) is given to us as the reference of a term.[8] The phrase is, of course,

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6 See, for instance, MacFarlane (2002). The difference regarding scope is evident. On the nature of logic, a basic point is worth noticing: Frege held that it is a substantial theory – a body of universal truths, which is constitutive of any reasoning and thought. On these and further features see Bar-Elli (1985). For Kant, logic is devoid of content, and does not consist of judgments at all (See Kant CPR, B191; Frege BS78, 85, 96).

7 I leave "Kantian strands" rather imprecise, for, as stated in the text, my main concern is with Frege's views, not Kant's, and I intentionally avoid entering into scholarly debates about Kant's exact view.

8 See his SR. The basic idea is central already in §8 of his 1879 *Begriffsschrift* (BS); for a detailed discussion see Bar-Elli (2006). It is operative again in FA, e.g. §67. This is the primary characterization of sense in SR. Later, in his BL, Frege talked of

Kantian. But the differences between them notwithstanding, it is not only the term but the general conception that things in the world are always given to us in particular ways (waiving for the moment the nature of these ways), which Frege inherited from Kant. Yet, he gave it a sort of a linguistic turn: A sense, for Frege, is a mode in which something (an object or a function) is being given to us, as this is expressed in the meaning of a linguistic expression referring to it. Hence, his conception of sense is heavily constrained by his elaborate theory of reference, on which it supervenes. Thus conceived, sense is yet an epistemic notion, where the cognitive value of statements lies.

All this has been much discussed.[9] What has been less discussed is a certain aspect of the notion of sense – its being a *justificatory* one: it lies, as I argue, at the basis of the justificatory enterprise in *justifying the axioms*, or the basic truths of a domain. And this is vital for Frege's notion of analyticity and for appreciating his view that arithmetic, though extending our knowledge, is still analytic. The issues concerned revolve around three main claims: 1) The notions of analytic and a priori apply only to justifications of propositions. 2) Axioms (basic laws) of logic are analytic, and those of geometry – a priori. 3) Deductive derivation (from truths) is a basic and paradigmatic form of justification, but when it comes to the axioms, which are not derivable from other truths, justification can take other forms. All three raise serious problems and have been challenged. Though I cannot discuss them in detail here, let me expand on them a bit.

Though the point has been quite surprisingly ignored or played down in much of the secondary literature, Frege is clear, in introducing the notion of analyticity in §3 of

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sense mainly as a constituent of thought. For a discussion of the relationships between the two see Bar-Elli (2001). Another characterization, which Dummett made prominent – as a "way of determining the reference", or a "route to the reference" – seems to me unhappy and has a slim basis in Frege; and a widespread explication of it, as a condition whose sole satisfier is the reference, seems to me strictly wrong in suggesting a predicative view of sense in which it is a concept or a property, which is opposed to Frege's explicit view.

9 Though some nuances in the above formulations are my own, the basic ideas have been much discussed at least since Dummett's (1973, 1981a). See for instance, Carl (1994); Bar-Elli, (1996).

Foundations of Arithmetic (FA), about its epistemic and justificatory nature.[10] A proposition is analytic, according to Frege, "if in carrying out this process [of finding a proof of it and following it up back to the primitive truths] we come only on general logical laws and on definitions" (FA, §3, p. 4). But this is not only implied by his "definition" of analyticity, but stated clearly as a governing principle:

> "**When a proposition is called a posteriori or analytic in my sense […] it is a judgment about the ultimate ground upon which rests the justification** [*Berechtigung*] **for holding it to be true**" (FA, §3, p. 3).

He emphasizes that "**where there is no justification, the possibility of drawing the distinctions** [between analytic, synthetic, apriori and aposteriori] **vanishes**" (ibid.). As he adds in a note there, Frege (perhaps wrongly) believed this to be also Kant's view. Some scholars believe this was part of his deep epistemological motivation in detecting the "ultimate grounds of judgments", which was basically Kantian,[11] or perhaps he wanted, at this early stage, to minimize the novelty of his approach and his departure from Kant, or he may have been simply wrong about Kant here.[12] In any case, later on Frege was quite clear about the difference between his notions of analytic/synthetic and Kant's, to the point he could say in the conclusion of FA, §88 (in a passage to which we shall come back towards the end) that arithmetic "extends our knowledge" and is synthetic in Kant's sense of the term, though analytic in his – Frege's.

Even granting the reducibility of arithmetic to logic on the basis of some definitions (to which I shall come back shortly), for understanding the analyticity of arithmetic we still face a problem about the epistemic status of the axioms of logic:[13] What is their

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10 This has been argued in detail in Bar-Elli (2010), 165-184, and in chs. 7-8 of my Hebrew book *(*2009).

11 See for instance, Weiner (1990), 54-55.

12 If Frege was right about Kant here (as some Kant scholars believe) this is another Kantian strand in his view, and it should not make much change to the main claims here.

13 In FA § 17 Frege considers a possibility that arithmetic is provable as a set of analytic conditionals whose antecedents are the logical axioms, just like geometry, with its axioms as antecedents. In both cases the conditionals themselves are

justification? In light of the above principle, they must have one if they are to be regarded analytic in Frege's sense, which very few scholars seriously doubt.[14]

Some people think the question is spurious because the axioms of logic are self-evident. However, besides the intrinsic problems of the notion, and even if we would, quite loosely, regard the self-evidence of a proposition as its justification, self-evidence in itself cannot be the end of the justificatory story. For the kind of justification relevant to analyticity we need to know in what way the axioms are self-evident. To appreciate the point one should note that the axioms of Euclidean geometry are presumably self-evident (as Frege himself thought) – they were considered for centuries the paradigm of self-evidence (definitely more so than the axioms of Frege's logic). Why then isn't this sufficient for rendering them, and whatever is logically derived from them, analytic? If being self-evident would suffice for the kind of justification Frege is alluding to, (Euclidean) geometry should be deemed analytic, according to Frege, just as arithmetic and logic. But Frege didn't think so – he thought that geometry, in spite of its axioms

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analytic, i.e. proved by the logical axioms, which are supposed to be the antecedents in the first case.

[14] Dummett, and many others, have taken the analyticity of logic and its axioms as almost a truism. Dummett accused Frege of carelessness in formulating his definition of analytic for that reason (See his (1991), p. 24). Burge, for one, claimed that Frege followed Kant in not regarding the axioms of logic to be analytic (Postscript in his (2005), p. 388). I can't get here into a detailed discussion of this point, and shall just state that in my mind there are good reasons to think (with Dummett and against Burge) that Frege did and should have regarded logic (and its axioms) to be analytic. However, what Dummett considered carelessness in Frege's formulation is due to his failing to see the justificatory nature of Frege's notion of analytic (the above quoted principle) and the fundamental role of the notion of sense in justifying the axioms (though not deductively). Dummett holds both that logical axioms are analytic and axioms of Geometry – a priori, and that the only form of justification is deductive inference, and axioms cannot be thus justified. But these two are incoherent with Frege's explicit principle (quoted above) that the notions of analytic, a priori, etc. concern the justification of a proposition. I expanded on this in Bar-Elli (2010) and in ch. 8 of my Hebrew Book (2009).

being self-evident, is not analytic. So, the presumed self-evidence of the axioms of logic cannot be the sole ground of their analyticity. It cannot exempt us from asking about the particular way in which they are self-evident, and the kind of justification they may have. Hence, the root of the difference between logic and geometry here must lie in the nature of the axioms and in the different ways in which those of geometry and of logic are self-evident or justifiable.[15]

One could perhaps retort here that Frege simply defines analyticity in terms of reducibility to logic. Hence, the analyticity of the axioms of logic, unlike those of geometry, is not in question. But surely, calling a judgment analytic because it is provable from the axioms of logic is not an arbitrary terminological decision. These axioms must be justifiable in a particular way that gives a rationale for this decision. This is enhanced by the above principle of FA. We are back then with our question: How can the axioms (i.e. each of them) be justified?[16]

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15 Robin Jeshion in her (2001) distinguishes a proposition being "*selbsverstandlich*" from its being "*einleuchtend*", both often translated as "self-evident", which she reserves for the second. The first refers to objective "foundational security", not in need of proof; the second – to rational, non-inferential justification. Both, she claims are constitutive of Frege's conception of axioms, and to his "Cartesian Conception", in which the second implies the first. She also argues that self-evidence is operative in the methodology by which Frege looks for basic truths and foundations. This search, she insists, is fallible – one can be mistaken in identifying a proposition as self-evident. This opens a way of explaining how Frege could have suspected the self evidence of an axiom, like law V of BL. I cannot discuss it here.

16 According to an influential trend in Frege's scholarship, the principles of logic cannot be justified. This is often connected with the view that there is no "Meta-logical perspective" to use Ricketts phrase (See also Weiner (1990). However, first, one can concede the latter without the former: justifying logical principles may be conceived as an inner logical enterprise. Secondly, the unjustifiability of logical principles is at flat contrast with their analyticity, given Frege's justifiability notion of analyticity (Bar-Elli (2010). Thirdly, besides some of Frege's formulations with which such a view is at odds, he does provide (verbal) justification for his logical

There is an austere sense of justification, namely derivation in *Begriffsschrift*, in which they cannot. But justification, for Frege is wider than this austere sense. In several places Frege explicitly recognizes a notion of justification wider than the deductive-inferential one:

> "Now the grounds which justify the recognition of a truth often reside in other truths which have already been recognized. But if there are any truths recognized by us at all, this cannot be the only form that justification takes. **There must be judgments whose justification rests on something else**, if they stand in need of justification at all. And this is where epistemology comes in" ("Logic", in *Posthumous Writings* (PW), 3).

This should apply also to the axioms, for there is an inherent connection in Frege between objectivity and justifiability: objective is what is justifiable, or plays a role in a justification.[17] Hence, in order to secure the objectivity of a domain (like geometry) its axioms do need justification, and their justification is therefore of this wider, non-deductive kind. There are further reasons to believe that Frege did hold this view.

In "17 Key Sentences on Logic",[18] article 13, Frege writes:

> "We justify a judgment either by going back to truths that have been recognized already or **without having recourse to other judgments**. Only the first case, inference, is the concern of Logic" (PW, 176).

So here again, much like in the previous quote, Frege unquestionably recognized this other (i.e. non-deductive) kind of justification. In these passages Frege does not yet say what this other, non-inferential way of justification is, though his talking of this other way as "epistemological" is an important hint. He did not yet have then the terminology for his notion of sense, and does not explicitly say that this other kind of justification has to do with the senses of the constituents in question. But later, in talking e.g. about geometry, equipped with his mature notion of sense, he gets much closer to explicitly expressing the connection between sense and the justification of the axioms. About

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axioms (Burge, 1998). Weiner (ibid) regards all these as "elucidations", which, she emphasizes, are not proofs in the system. She is of course right in that but seems to ignore the claim at issue, namely that proof is not the only form of justification – there are justifications, which are not (deductive) proofs in Frege.

17 This is argued in detail Bar-Elli (2010).

18 The editors of the *Nachlass* date it not later than 1906. Dummett suggested it is a much earlier work, see his *Frege and Other Philosophers* (1991a), pp. 66, 77.

(Eucledean) Geometry, it is often realized that Frege held a Kantian view, according to which geometry is synthetic a priori. What is less often realized is that for Frege (probably unlike Kant) this concerns only its justification. The syntheticity of geometry consists in the fact that its justifiability – the way the geometrical truths, basically the axioms, can be justified – requires intuition (*Ansschauung*), which is at least part of the way in which geometrical objects and concepts are given to us – their senses.

In a pivotal point of FA Frege considers the equivalence "The direction of *a* is the direction of *b* if and only if *a* is parallel to *b*" ((D(a)=D(b) ≡ a//b). He rejects reading it as defining parallelism in terms of directions. And the reason he gives is that such a definition does not respect, as it should, the ways things are given to us: "Everything geometrical must be given originally in intuition" (FA, 75).

Likewise, and even more to the point, he talks in a similar vein about the axioms and says that the axioms are justified on the basis of the senses of their constituents – the ways their objects and concepts are given to us:

> "So long as I understand the words 'straight line', 'parallel' and 'intersects' as I do, I cannot but accept the parallels axiom…**Their sense (***Sinn***) is indissolubly bound up with the axiom of parallels**" ("Logic in Mathematics", PW, 247).

In "Foundations of Geometry I", CP 273/319-284/375, of 1903, after claiming that axioms, including those of logic, are certain without being provable, Frege says "**Here we shall not go into the question of what might justify our taking these axioms to be true**" (273). This seems to imply that he could go into this question, and there is something that can serve as such a justification. He doesn't say that there isn't, or that one could not go into the question, but only that he wouldn't do it on that occasion.

He then says that "In the case of geometrical [axioms], intuition is generally given as a source", and later: "Never may something be represented as a definition if it requires proof **or intuition** to establish its truth" (275). Again, this clearly implies that there is something except proof that can justify or establish the truth of a thought. And the context of his polemics with Hilbert about regarding the axioms as "implicit definitions" suggests that he was thinking here primarily of axioms. Intuition and "basic facts of intuition" are repeatedly presented in the sequel as the source of the validity or justification of the geometrical axioms.

Towards the end of his late article "Compound Thoughts" [*Gedankegefuege*] Frege says: "for the truth of a logical law is immediately evident from itself, i.e., **from the sense (***Sinn***) of its expression**" (405). And in a piece he probably wrote in the last year of

his life Frege wrote: "**From the geometrical source of knowledge flow** (*fliessen*) **the axioms of geometry**" (PW, 273). The axioms then "flow" from something; they have epistemic grounds or justification. And this, I suggest, is basically the ways the geometrical things they are about are given – the senses of their terms.

I therefore surmise that Frege thought that axioms, though unprovable, are justifiable by detecting the source of their knowledge. And, though he doesn't say so in these very words, a view about this other, non-deductive form of justification can be gathered from various scattered remarks of his, and is, any way, in conformity to basic lines of his thought. A sketch of its general outline can be put as follows: The justification of axioms, in geometry, as well as in logic, is given in terms of the senses of their constituents – the ways the things they are about are given to us. These ways are different in the two cases: in geometry they consist of special (spatial) intuition; in logic, they consist of basic features of our ability to think and reason. Calling the latter, and not the former, analytic is therefore well motivated. Grasping the sense of the constituents of an axiom is not the only way of justifying it and is not sufficient for such a justification. For first, such grasp may be incomplete and hazy. Secondly, the "network of implication relations" of the propositions concerned must also be taken into account (PW 205). There is of course a deep connection between the two, and yet one cannot expect even a complete grasp of a sense to cover all the pertinent implication relations in which it is embedded.

THE ABILITY TO RECOGNIZE OBJECTS

The above means that an epistemic and justificatory notion of sense – a mode in which something is given to us as the reference of a term – is central to establishing the analyticity of logic, and hence – of arithmetic.[19] This brings us to another important

[19] This option is absent in the discussion of Frege's views on the justification of basic logical laws in Weiner (1990), ch. 2. Moreover, Weiner is unclear on whether primitive logical laws are justifiable: On the one hand, she recognizes that "not all justification, on Frege's view, is inferential" (p. 61), and "the justification of a primitive logical law is evident from its content" (though not by the mental act of considering it, p. 78), while on the other she also says that "primitive logical laws cannot be justified" (p. 77).

Kantian strand in Frege's view, the above differences between them notwithstanding. Kant thought that recognizing an arithmetical truth requires a special intuition, *Anschauung* (hence, their synthetic nature). Frege's view, as portrayed above, should not be deemed opposed to that (though he wouldn't use the term intuition (*Anschauung*), and the nature of the intuition concerned is different): the justifiability of arithmetical (and logical) truths depends on that of the axioms – ultimately on those of logic; and this, construed in terms of the senses of their constituents, as sketched above, **depends on there being (**logical**) objects whose modes of presentation to us these senses are**.

Intuition is required, on Kant's view, for a representation of an object. In general, objects are recognized by concepts and intuition – the former responsible for the "unity in consciousness" of the manifold of representations; the latter – for their singularity (CPR, A103-110; cf. also B376-7). A characteristic mark of his view is that it depends on sensibility, which is Kant's general term for the mental capacity perceptive to the ways objects are given us: "Without sensibility no object would be given to us" [*Ohne sinnlichkeit wuerde uns kein Gegenstand gegeben*].[20] This general capacity is activated by sensations, which, as Kant makes clear, are required for what he calls "empirical intuition"[21]. Kant also recognizes what he calls "pure intuition" of the pure forms of intuition, namely space and time. Some interpreters find it profitable to separate these two factors, and regard intuition as required for representing objects in general, whether it depends on sense perception and sensibility or not.[22] I shall not delve here on this much discussed issue in Kant.

For my concerns, the important point to note is that Frege's view of arithmetic also requires such a capacity, in spite of his rejecting what he took to be Kant's view that it depends on sensibility – whether empirical or pure. In a wide sense of "intuition" as a capacity of recognizing objects (without the restriction to sensibility) it therefore requires intuition. Frege devoted much space and effort to establishing that numbers are objects, and he regarded the question of how logical objects (in particular, numbers) are given to us as central to [the philosophy of] arithmetic. Sure enough, a crucial thesis of Frege's is

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20 Kant, I.: *Critique of Pure Reason*, 1933 (CPR), A51/B75.

21 See for example, Kant, op. cit. A 19.

22 See, for instance, Hintikka, "An Analysis of Analyticity", in his (1973), 123-149, p. 145, and other articles referred to there.

that objects (e.g. numbers) can be given us by logic and reason, independently of sensation and space and time.[23] And yet, as stated above, a general capacity to recognize objects and ways they are given to us is required by Frege's conception of the objectivity of logic and arithmetic. Whether this cognitive ability to recognize objects is called "intuition" or not is of lesser importance. Whether it depends on sensibility – as Kant proclaimed and Frege denied – is a more important and substantial issue. However, it should still not blind us to a main point of agreement, namely, that for both Kant and Frege, **our knowledge of arithmetic depends on our ability to recognize objects and their existence**.

When this is properly appreciated, another Kantian strand in Frege's thought emerges: the dependence of the objectivity of arithmetic on our ability to recognize objects. This is not a trivial similarity – even among logicistic approaches it is a distinctive mark of Frege's: The analyticity of arithmetic depends on that of logic, and on the justifiability of its axioms. This is accomplished in terms of the senses of their constituents. Sense, for Frege, is a sense of something – of an object or a function.[24] It is

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23 The point is clearly made e.g. in FA §89, and earlier in §23 of *Beriffsshrift*.

24 This is a notoriously debated point. Some of Frege's formulations suggest that there are referenceless senses. But this, I believe, is not his better and considered view. His principal characterization of the sense of a term is a mode in which its reference is given to us, and it is hard to make sense of this where there is no reference. Moreover, a thought, which is the sense of a proposition, and is built up by the senses of its constituents, is essentially true or false, which according to Frege is impossible when some of its terms lack reference. And indeed, in his logical language, Frege takes care to ascertain that all terms have references, and repeatedly claims that this must be the case in any "scientific" language. Examples he gives from natural language of names lacking reference (e.g. Odysseus) are of fictional characters in artistic, non-scientific, contexts, and he was an extreme subjectivist about art, claiming that it does not express genuine thoughts, but "apparent thoughts" (*Scheingedanken*, sometimes translated "mock-thoughts"), and he sometimes speaks of such names as "apparent names" (*Scheinnamen*); Cf. Bar-Elli (1996), chapter 3.

a mode of its being given to us as the reference of a term. A function, in turn, depends on objects, and conceiving a function depends on our ability to recognize objects.[25]

Moreover, it was this Kantian conviction that led Frege to insist on the existence of logical objects, which eventually led to the contradiction (the so called Russell paradox) and to what he regarded as the failure of his logicistic project. The above conviction may at least partially explain Frege's "obsession" with logical objects (truth-values and extensions of functions). For, without them it is hard to see what the senses of the constituents of the logical axioms could be senses of, and deprived of these, we have a poor notion of the justification and analyticity of the axioms of logic (and arithmetic).[26]

Logical axioms are conceived by Frege as universal truths, which are construed, in general as (second-order) predication on functions. A function for Frege (including concepts, which are functions to truth-values), though real and objective, is a particular way of connecting objects – connecting the arguments of the function to its values. This is its whole essence and "being".[27] The notion of a function therefore supervenes on that of object and talking or thinking of functions supervene on the ability to recognize objects. Throughout his career Frege maintained that functions are grasped only through their linguistic expressions. In §9 of his early *Begriffsschrift*, still lacking his sense/reference distinction and talking in terms of "contents" (Inhalt) of expressions, Frege was unable to express the crucial distinction between a content and a particular way in which it is given. He was therefore almost forced to identify a function with an incomplete expression.[28] Later, in his mature position, beginning

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25 In his (2004) P. Sullivan says that "Frege's claim must be some version of this idea: that objectual bearing consists in structural features internal to the nature of thought, and "unfolded" by the laws of logic" (704). The exclusive appeal here to "structural features" is I think too restrictive, for, as explained in the text, a thought is a sense (of a sentence), and its objectual bearing is rooted also in that of Frege's notion of sense. Sullivan may be excused for underrating this, since he focuses, as he says, on *Begriffsschrift*.

26 This has been argued in detail in Bar-Elli (2001).

27 This is emphasized and largely expanded by Dummett. See, e.g his (1973, 1981a), ch. 8.

28 See on this Bar-Elli (2006), where some alternative views are examined.

with "Function and Concept" (FC), he was clear that functions belong to the realm of reference and are real and objective. However, he still maintained that they are unsaturated and graspable only through the incomplete linguistic expressions that denote them. In explaining the function denoted by a functional expression like $(2+3x^2)x$ in *Basic Laws of Arithmetic* (BL) Frege says:

> "The essence of a function is revealed rather in the connection established between the numbers whose signs replace 'x', and the numbers that then appear as *Bedeutungen* of our expression… The expression of a function is incomplete, unsaturated (*ungesaettigt*). The letter 'x' merely serves as a place-holder for a numeral to complete the expression…" (BL, §1).

Though Frege uses here algebraic examples, he expands the concept of function, in terms of both arguments and of values, to include concepts and relations, which are thus conceived as functions whose values are the two truth-values (e.g. ibid. §2). This is a main point in (FC) and has also its root in §9 of *Begriffsschrift*. Hence, concepts also have their essence and "being" in their applying to objects. Frege repeatedly emphasized the idea by saying that they are "**essentially predicative**" (e.g. "Concept and Object" (CO), 182/193; Letter to Russell, 13.11.04, PMC, 161).

In sum, Frege's conception of functions incorporates all the following theses:

1. Functions are real objective entities in the world (in the realm of reference).
2. Functions are not objects (including extensions or sets) – there is a categorical difference between functions and objects.
3. The essence of a function consists in the relationships between objects (noting that relations are themselves functions), to which belongs the idea that functions are essentially incomplete.
4. For grasping the sense of a function, a capacity of recognizing objects is required.
5. A function can be known only (or at least typically) through grasping senses of linguistic expressions referring to it.

Hence, when logical axioms are conceived as universal propositions – as predications on functions – this should not detriment our claim that they are intrinsically connected to our capacity to recognize objects, because this capacity is essential for our notion of a function. The connection here is admittedly more remote than in a simple thought about a

particular object, but it is still valid and important to be noticed.[29] In talking of this intrinsic connection I do not mean to claim for the priority of one side over the other – rather, they go hand in hand. In grasping a thought we grasp its constituent senses, which are modes in which their references are given to us. These references are either objects or functions. And when they are functions their recognition ultimately depends on that of objects. And with respect to the logical axioms in Frege's system, the functions concerned are functions of logical objects – truth values and extensions.

One of the most obvious features of the course of argument in FA is that Frege sees it necessary to establish that "**Every individual number is a self-subsistent object**" (p. 67). He then asks the crucial and typical question: "**How then are numbers to be given to us**" (§62). This is the starting point and the pivotal move in his developing his view that numbers are logical objects that are given to us, or definable, by logic alone. He shows this first for the concept Number (§68) and then for the individual numbers (from §74). This general strategy is maintained in BL.

It has been argued (for instance, in Bar-Elli (2001), that for Frege, logical objects – the truth-values and extensions of functions – are indispensible for reasoning and thinking and are required by what he regarded as irresistible logical principles, like axiom V of BL (already recognized in FC) and the axioms of truth-functional logic.

In "On Sense and Reference" (SR), after establishing the True and the False as the reference of sentences, Frege writes:

> "Every declarative sentence concerned with the reference of its words is therefore to be regarded as a proper name, and its reference, if it has one, is either the True or the False. **These two objects are recognized, if only implicitly, by everybody who judges something to be true**..." (33/63)

It is also argued there that examining the justifications Frege gives to the axioms in BL (§18) suggests that the axioms of truth-functional logic are not only truths about the truth-values (or functions over them), but they express on this view aspects of the ways the True and the False are given to us as logical objects.

Frege ends the appendix II, (*Nachwort*) to BL by stating: "**The prime problem of arithmetic is the question, In what way are we to conceive logical objects, in**

[29] On thoughts about objects and the significance of the notion of about in Frege, see ch. 7: "Reference and Aboutness" of my (1996).

particular numbers?" (143). ("Way of conceiving" is one of the expressions Frege uses for his notion of sense, and it is virtually synonymous here with "way of being given".)

Similar remarks accompany the introduction of value-ranges in FC and BL. And in a letter to Russell of 28.07.02, even after realizing the trouble into which axiom V leads, Frege wrote":

> "But the question is, How do we apprehend logical objects? And I have found no other answer to it than this, We apprehend them as extensions of concepts, or more generally, as ranges of values of functions" (PMC 140-1).

So, the recognition of logical objects amounts not only to realizing the truth of the corresponding axioms, but also to a particular way of this realizing: to a special construal of these axioms as being "about" those objects, and of their "self-evidence" as being grounded in their expressing features of the senses (*Sinne*) of their constituents – the ways these (logical) objects are given to us.

Hence, grasping the logical axioms, and logical truths in general, requires a capacity of recognizing objects, which in a wide sense of the Kantian terms involves intuition (*Ansschaung*). The fact that in many cases this is a very special kind of intuition, which constitutes our ability to think and to reason, is of course important and marks the location of an important difference between them,[30] but it should not blur the substantial common ground: The notion of objectivity and our ability to have objective judgments – arithmetical ones included – depend on there being objects given to us in particular ways. Frege's repeated insistence on these being "objects" means that the capacity to recognize logical objects, which is necessary for grasping the sense of the logical axioms, is a sub-species of the general capacity to recognize objects. This makes the comparison with Kant's view the more pertinent, and this is true even independently of Frege's more extreme position that a function should be definable for all objects. This then is another Kantian strand that interweaves into the Fregean view of the basis of the justification of arithmetic – hence of its analytic nature. And note – "analytic" is here in Frege's sense.

We thus see that Frege was constantly concerned, both in his conception of logic and of arithmetic, with the senses of logical objects – the ways they are given to us.

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[30] It is arguable that in Kant intuition is also a pre-requisite for our ability to think, but the meaning and the argument for this are different than in Frege, where it is much more straightforward. In any case, such a reading of Kant would strengthen this Kantian strand in Frege.

Granted this, one could still wonder why Frege should presume the existence of logical objects at all. We have detected at least two lines of thought in Frege for establishing the need to recognize logical objects. The first consists in establishing that logical propositions are about, or concerned with logical objects. This first line of thought is supported by three main arguments: the first is that once it is realized that Truth and Falsity are the references of sentences, they must be recognized in any grasp of a proposition – in any serious act of thinking a thought. The second is based on the general functional conception of the truth of a judgment, i.e. that it is the satisfaction of a function, basically by objects. The third is that basic truths of logic are concerned with logical objects – either directly (in being about truth values or extensions) or, more remotely, in being second-order predications about (logical) functions.

The second line of thought is that logical axioms, in order to be objective and analytic, must be justifiable. Their justification cannot be founded on deductive inference, but must be of another kind. This other kind consists mainly of the justificatory nature of their constituent senses. And these senses, once again, must either be directly senses of logical objects, or be senses of logical functions, whose grasp ultimately depends on that of logical objects. In either case this kind of justification carries with it the need to recognize logical objects.

Beams and Seeds - Fruitful Analytic Definitions

But the point has also to do with the nature of definitions and their role: for Frege, analytic, let us remember, is what is justifiable by (or reducible to) logic and definitions alone (FA, §3). These definitions (of the basic arithmetical terms) cannot be philosophically unconstrained. If they were, any consistent (first-order) theory would be analytic, for any such theory is reducible to logic by some set of definitions. The definitions of Frege's reduction of arithmetic to logic, moreover, are not only what he calls "constructive" or stipulative definitions of new terms. Rather, they are what he calls "analytic" (*zerlegende*) definitions of terms in use, whose meanings are partially and perhaps dimly recognized.[31] Hence, these definitions must satisfy some constraints.

[31] See "Logic in Mathematics", in his PW, 227/210-211. Some authors ignore this important notion. Weiner (1990, ch. 3) for instance not only ignores it but adopts such an austere and restricted notion of analysis that makes it incoherent, which shows that it can't be Frege's notion.

What constraints? This brings in a wide and complicated subject, on which I cannot dwell here.[32] Let me just hint at one point. Geometry (Euclidean geometry) has a model in arithmetic. That is, there are "definitions" of the geometrical terms, by which the axioms of geometry would be truths of arithmetic. Hence, if definitions would be left unconstrained, since arithmetic is reducible to logic so would Geometry, which should then be deemed analytic (in Frege's sense). All this was of course well known to Frege, and yet he rejected the conclusion. The way Frege would have blocked this move is, I guess, by philosophical constraints that should be imposed on the "definitions" of the geometrical terms, for such modeling of geometry in arithmetic would not satisfy these constraints. This idea is important for understanding a central move in the course of the argument of FA, to which we have already alluded, where a certain provisional way of defining expressions of the type "the number of Fs" is rejected. In explaining the rejection Frege appeals at a certain stage to a geometrical example in which the equivalence $D(a)=D(b)\leftrightarrow a//b$ [the direction of a is the direction of b iff a is parallel to b] is taken as a definition of parallelism. Frege rejects this because "**this is to reverse the true order of things**" [*der wahre Sachverhalt damit auf den Kopf gestellt wird*] "**For everything geometrical must be given originally in intuition**". Parallelism is given in intuition, Frege thought; directions are not (FA, §64, p. 75). Therefore one cannot define the former in terms of the latter. The general lesson to be learned – a lesson without which FA and in fact his whole project, cannot be understood – is that the pertinent definitions are philosophically constrained, where a dominant consideration here has to do with the ways the things concerned are given to us.

As stated above, I cannot delve here on the nature of the required constraints on analytic definitions. What I would like to suggest here is that the fact that the pertinent definitions are "analytic" – are of terms in use whose meaning is partially grasped – introduces another Kantian strand into the picture. To put it roughly, an analytic definition explicates the meaning of a term in use, and the conclusions logically drawn from such definitions are, in some sense, contained in the definitions, hence – in the meanings of these terms. Let us see a crucial passage here: Towards the end of FA, after explaining that the poverty of the logical structure of traditional (and Kantian) definitions

[32] This is expanded in ch. 7 of my ibid. Hebrew Book (2009), above note 10

is artificially restrictive in using the boundary lines marked by the old concepts (of the definiens), Frege continues about his own definitions:

> "But the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all […] **The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic**; and yet they can be proved by pure logical means, and are thus analytic. The truth is that **they are contained in the definitions** (*in den Definitionen enthalten*), but as plants are contained in their seeds, not as beams are contained in a house. Often we need several definitions for the proof of a single proposition, which consequently is not contained in any one of them alone, yet does follow purely logically from all of them together" (§88, p.100-1).

This passage is particularly rich and illuminating. Frege in fact makes here clear that the issue between him and Kant is not so much about the nature of arithmetic as about the notions of analytic and synthetic, and about the nature of definitions. Frege claims, in agreement with Kant, that arithmetic, i.e. the conclusions logically drawn from the definitions concerned, extends our knowledge, and is synthetic in Kant's sense of the term.[33] His claim that it is yet analytic displays his awareness that they were using these terms differently. And yet, Frege expresses here also his sophisticated version of Kant's famous "containment condition" of analyticity: Arithmetical truths are contained in their concepts (or their definitions) as plants are contained in their seeds. Thus, Frege subtly trades here on two central Kantian themes: on the one hand, arithmetic extends our knowledge (and is, in Kantian terms, synthetic), while on the other, it is in some sense contained in the meanings of its terms (and is thus analytic).

It might seem that Frege's "containment", which is a relation between judgments, is very different from Kant's, which is a relation between concepts. But this doesn't seem to me to go very deeply, for the containment of e.g. 'round' in 'circle' is no different than the containment of 'x is round' in 'x is a circle'. And when Frege speaks of analytic truths as derivable from definitions alone he sometimes speaks of them as being derived from a concept. About the laws of identity, for instance, he says: "**As analytic truths they should be capable of being derived from the concept alone** [*aus dem Begriffe selbst*]"

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[33] The basic idea is repeated at the end of § 91, where he says that "**propositions which extend our knowledge can contain analytic judgments**".

(FA §65, p.76).[34] Moreover, Kant's notion is not strictly a relation between concepts, for he thought of syllogisms like "If all humans are mortal and all Greeks are human, then all Greeks are mortal" as analytic.[35] It is plausible to assume that he would regard the conclusion of a deductive inference, at least in simple cases like such syllogisms, as "contained" in its premises. Hence, the difference between construing containment as a relation between concepts or judgments is not crucial here.

A definition in itself, according to Frege, is not a statement that can be analytic or synthetic.[36] In the above passage from the conclusion of FA Frege speaks of the fruitfulness of (his) definitions – a topic which recurs in his writings. The notion of fruitful (*fruchtbar*) concepts (definitions) and its metaphorical presentation as setting new borderlines (carving new areas of reality) is central already in the early "Boole's Logical Calculus and the Concept Script" (BLC) (in PW, see especially pp. 33-35).

It is not easy to be precise on what Frege meant here, but some points seem clear. As Frege repeatedly claimed the definition must be operative in proofs of significant theorems in the field. This is a sign of the definition's analyzing the content "at its real joints", where the structure and order of dependencies in the field concerned, and connections between propositions and concepts within it are manifest. As Frege says: "The insight it [a genuine definition] permits into the logical structure… is a condition for insight into the logical linkage of truths" (*Collected Papers* 302). A characteristic facet of such analyses is that they are achieved in terms of (nested) quantification and variables. The Weierstrass definition of the continuity of functions served as paradigmatic example: F(x) is continuous at a iff $(u>0)(Ev>0)(x)((x-a)<u\rightarrow(f(x)-f(a))<v)$.

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34 This, by the way, in contrast to Burge's view that Frege did not regard logical principles to be analytic (see note 14 above). There are other reasons to doubt Burge's position here, e.g. that Frege was aware that the same proposition can be an axiom in one system, while provable in another, logically equivalent one (e.g. *Begriffsschrift* 29, PW 206), but it is hard to adopt this kind of relativity to the notion of analyticity. I shall not go into the details here and just assume (with Dummett and many others) that Frege would regard logical axioms to be analytic, and geometrical axioms to be a priori.

35 See C.D. Broad (1978), p, 4.

36 Cf. *Begriffsschrift*, §24; FA §67, p.78; PW p. 224-6/207-9.

The fruitfulness concerned is another expression for the extending knowledge of which Frege speaks in the above passage. This, as he says there, is usually a matter of deriving a proposition from several premises and definitions. But it may also be a derivation from one sentence, if the proof of the conclusion is not trivial and immediate.[37] In any case, as e.g. Tappenden (ibid.) emphasizes, the extension of knowledge concerned is not merely psychological: it is objective and concerns the objective order, connections and structure of the (mathematical) field.

We said before that strictly, a definition, on Frege's view is not a statement. There are, however, two points to note here in which a definition may give rise to a significant statement: (1) The definiendum, the term defined, may be a significant term in use, whose meaning cannot be disregarded. The definition in that case is very close to a statement analyzing this meaning or use in a way that must conform to it – what Frege calls (in "Logic in Mathematics", ibid.) explication (*Erleuterung*).[38] (2) The definiens may, and in most interesting cases would, express or catch a "new" concept – "new" in the sense that, relative to its constituents, it carves up reality (the content) in a new way.

In BLC and FA Frege emphasizes that a definition is not just a conjunctive or disjunctive combination of characteristic properties (*Merkmale*), but that "**every element in the definition is intimately, I might almost say organically, connected with all the rest**" (FA, §88, p. 100). Evidently, Frege conceived of this "organic connection" in terms of logical structure and in particular – quantification and bound variables.

Suppose, for example, you have two concepts A and B. You can define on their basis a concept "Cx ≡ (Ax & Bx)", which would apply to anything that is both A and B. But this, Frege claims, would be trivial definition, not forming a really new concept, because it uses and is restricted to the "boundary lines" (if we present concepts by bounded areas) of the old concepts, A and B. On its basis, statements like "All Cs are As" are (trivially) analytic both in Kant's and in Frege's sense. But one may also define a new concept that, by its logical structure, would "delineate a new area in reality", not confined to the boundary lines of the old concepts in terms of which it is defined. The same point,

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37 See on this Tappenden (1995); on this particular point see p. 435, where he ascribes the idea to Dummett's (1981), p. 300.

38 Cf. Kambartel's introductory chapter to Frege's *Nachgelassene Schriften*, XVII-XXV.

with the same metaphor of drawing boundary lines, is central in Frege's early BLC, and it applies not only to conjunction but to any combination of Boolean operators. Frege protested against Kant (and Boolean logicians) that he thought only of definitions of the first kind, which is a severe and artificial restriction, while the interesting definitions in logic and mathematics (including his own in *Begriffsschrift* and FA) are of the second kind. Take for example Frege's definition of **successor** in a series: The definition is couched in terms of the notion of hereditary property, which, in turn, is defined in terms of the general notions of a property and of a relation or function: G is **hereditary** in a series defined by f iff any object, which bears f to an object which is G, is also G [H(G,f) ≡ (x)(Gx & f(x,y) →Gy)]. Using this, Frege defines successor thus: **y succeeds x** in a series f, iff y has any hereditary property in the series that any object bearing f to x has [S(y,x) ≡ (z)(G)((H(G,f)&f(z,x) →Gz) →Gy)].[39]

When specific properties and relations are concerned, one can see what Frege means by saying that the definition of successor "carves a new area in reality". Suppose, for example, you have two properties – wise and tall – and a relation – being the son of-. Suppose moreover that being tall is hereditary with respect to the relation being son of-. One can then define successor as above, and the idea of successor would be "new" with respect to this modest basis; it would "draw a new boundary line in reality". There is a big difference between the boundary lines of the properties in the base and the one set by the definition. And this difference is traversed by logic – by the logical structure of the definition – a structure which also exhibits and gives us an insight into implication relations between propositions, "the logical linkage of truths" (CP, 302). Frege can then prove results like: "whatever bears f to a successor of x is itself a successor of x", which are "new" and extend our knowledge.[40] Hence, they are synthetic in Kant's sense (and yet analytic in Frege's). This then is one element in the "containment" Frege speaks about, by which the other Kantian strand I mentioned emerges: the result is contained in the defined concept as plants are contained in their seeds – logic and the logical structure of definitions is assimilated here to the biological process of growth. And as natural it is for

[39] Following current usage I use "successor" for Frege's "follower" [*Folgenden*], in distinction to "immediate successor" [*Naechstfolgenden*].

[40] Cf. Frege's slightly more complicated example at the end of FA §91.

us to say that a plant is "new" relative to its seed, so, given the biological process of growth, it is to say it is still "contained" in its seed.

But there is another element in Frege's talk of "containment". A definition, as noted above, may be "analytic", i.e. of a term in use whose meaning is grasped (if only dimly and partially) independently of the definition, as is the case in most Fregean definitions of arithmetical terms. Being "analytic", what is expressed by the definition, even if it draws new boundary lines, is in some way "contained" in the sense of the term defined. The above definition, for example, not only shows that the idea of successor is in some sense contained in that of property and relation (as plants in a seed), but being an analytic definition it also must display the defined notion of successor and conclusions drawn from it as being "contained" in the pre-theoretic idea of succession. This should not be seen as casting any doubt on Frege's valid point about the fruitfulness of his definitions and about their carving new areas in reality. But "new" here is in respect to the old concepts in terms of which the definition is couched. This doesn't conflict with the claim made here that this "new" area in reality is already "contained" (in the sophisticated Fregean version) in the sense of the defined term as used independently of the definition. Sure enough, a Fregean definition (or explication) goes beyond and deeper than its pre-theoretic "origin", but it is not entirely new relative to it: it is, after all, a definition and explication of it. Hence, there is no absurdity in a sort of a Kantian claim that even with regard to such Fregean fruitful definitions, say that of successor, whatever is proved by means of it is already, in some way, contained in the sense of "successor".

"In some way" is here illusive – there is an important difference between the way Kant thought about this containment, and the way it can be modified to apply to Frege. First, as Frege himself put it in the above quotation, there is the containment of beams in a house, and that of plants in their seeds. Kant's was the first; his (Frege's) – the second. Even when restricted to stipulative "constructive" definitions, Frege's notion is much richer and logically more sophisticated. Secondly, it also gets cleared of whatever psychological overtones one may hear in the Kantian notion (in terms of what one actually thinks in grasping such a truth). Thirdly, there is a great difference between the containment in a Kantian definition, and that in a Fregean explication. And yet, it is significant that Frege found it appropriate to stick to the ideas of analysis and containment here. Deeply, this is also a Kantian strand.

Chapter 9: Conceptual Analysis and Analytical Definitions in Frege

Although Frege's interests were obviously focused in logic and the philosophy of mathematics (mainly in the foundations of arithmetic) his way of handling them is marked by broad and deep philosophical views, and by being involved in some general philosophical positions and doctrines, sometimes only hinted at and sometimes more fully developed. These, without in any way depriving Frege's technical achievements of their significance, may outlast the technical validity of his logistic project and are independent of it.

A central such doctrine, which in fact is the heart of what Frege thought he was doing, is his notion of **logical analysis** – its goals, its importance, and the constraints governing it. This is often construed as **conceptual analysis**, which is a very common notion, also in the Fregean literature, but it is often involved with some confusion, and some preliminary remarks seem here in order. For the mature Frege, just as proper names have reference (*Bedeutung*) and sense (*Sinn*), so do predicative expressions such as '... is a number', '... is a circle', '... is a man', '... is bigger than ---' etc. The references of these expressions are functions from objects to truth-values, and this for Frege is a very general category of entities including concepts and relations. 'Conceptual analysis' is not Frege's term. Moreover, its wide use notwithstanding, strictly, from a mature Fregean point of view, it is inappropriate: Concepts are references of (one-place) predicative expressions, they are functions from objects to truth-values, and as such have no structure with determinate components and cannot be analyzed; only their senses can.

One of the main reasons for that is that in general, the reference of a part (of a complex expression) is no part of the reference of the whole. Taking, for instance, 'the capital of Sweden' as a complex name referring to Stockholm, Sweden is no part of the capital of Sweden, and, assuming it to be the most populated city of Scandinavia, Scandinavia is also no part of it, nor are the references of numerous other terms in expressions referring to it; Likewise, 5, 3 and the minus function are no parts of the number 2, which is the reference of '5-3'; similarly for concepts and functions in

general, as can be seen by considering several definitions of a concept, e.g. a circle.[1] This is no small matter, for it is crucial for understanding how different expressions with different constituents can refer to the same thing (object, or function), which is vital for science, mathematics and thought in general.[2] An important corollary is that although the references of the parts determine that of the whole, the reference of a whole and that of a part do not determine the references of the other parts. For example, the references of 'a' and of 'F' determine that of 'a is F', while the references of 'a' and of 'a is F' do not determine the reference of 'F'.

This is of course entirely different with regard to sense (*Sinn*): In general, the sense of a part is a part of the sense of the whole. Senses have structure and determinate constituents (though there may be exceptions to this).[3] What we usually

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1 See Frege's 'Notes for Ludwig Darmstaedter' (1919), PW 255/275, (FR 364-5), though there are some unfortunate formulations in earlier writings. One occurs already in SR (150/36 in FR): ' The whole *Bedeutung* and one part of it do not suffice to determine the remainder'. His intention is clearly as stated in the text, namely that the *Bedeutung* of the whole (expression) and that of one part of it do not suffice.... This is evinced also by his own explanation in the same paragraph that parts and analysis are discerned in the sense, not in the reference by itself. In his 1981a, Dummett rightly regards the idea that the reference of a part (of a functional expression) is part of the reference of the whole 'manifestly absurd' (265).

2 This central strand in Frege found an emphatic conclusion in Wittgenstein's doctrine in the *Tractatus* that 'Objects are simple' (2.02), and that 'Every statement about complexes can be resolved into a statement about their constituents...' (2.0201), in which the intention I surmise, is 'Every statement that is apparently or allegedly about complexes...' for the upshot of the doctrine is that there are no complexes. In relating it to Frege, one should remember that Wittgenstein rejected Frege's doctrine of sense. I shall not go into this here.

3 Facts, for Frege, are true thoughts – senses of true propositions. In the Appendix to Part I of *Philosophical Grammar* (tr. A. Kenny, Oxford 1974) Wittgenstein argues that facts are not complexes. This is probably directed mainly against Russell, who persistently construed facts as complexes. In connection to the problem of complexity he writes: 'To say that a red circle is composed of redness and

call conceptual analysis is therefore primarily not an analysis of a concept but of its sense. Derivatively it can be regarded as of a concept, but only when conceived under a sense. Though, for simplicity, I speak here of concepts, their referring predicates and their senses, much of what I say here applies to other kinds of reference (objects, functions) and to their senses, in particular thoughts, which are senses of propositions and are often what is analyzed. Keeping this in mind, the intention in my using 'conceptual analysis' here, in accord with its wide use, is I hope sufficiently clear. It also explains why much of what we shall say about analysis is focused on the notion of sense. Presumably, for these reasons Frege, as noted above, did not use 'conceptual analysis'. He spoke of 'logical analysis', but 'logical' is used here in a wide sense, and with the above reservation I shall use both terms interchangeably.

It seems that a minimal requirement of a logical (conceptual) analysis is that analysandum (A) and analysans (B) refer to the same concept. 'The same' is problematic here because concepts are categorically different from objects, and identity (sameness, equality) strictly applies only to objects. Hence we need a similar equivalence relation applicable to concepts – call it simply 'equivalence'. Concepts (of first level) are equivalent according to Frege when they have the same extension – when they apply to the same objects. This is the criterion of their equivalence.[4]

Now, two main questions arise regarding analysis.

1. Does any change in the extension of A and B amount to such a difference in the concepts that the analysis would be incorrect? Can't we endorse an analysis, say of number, which results in some change in the extension of A, particularly where, like e.g. with the naturals, the integers, or the rationals, the extension of B includes that of A? Frege was notoriously quite rigid on this question and thought that any change in extension is a change in concepts. We shall not deal with this question here.

2. Assuming that in an acceptable analysis A and B refer to equivalent concepts, unless the case is trivial they obviously have different senses. What should be the relationship between these senses to render B as an appropriate analysis of A? This is

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circularity, or is a complex with these component parts, is a misuse of these words and is misleading (Frege was aware of this and told me)' (200).

[4] This is stated already in FC and is implied by the famous axiom V of Gg. Frege elaborates on it in 'Comments on Sense and Reference', PW 118/128-125/136 (in FR, 172-180).

obviously a central question – perhaps the central question – of the nature of analysis, for obviously not any expression referring to an equivalent concept would be regarded as a conceptual analysis of A. Further constrains are required. Some were traditionally recognized and are not particularly Fregean, as for instance the requirement of **non-circularity**: Analysans should not include or presuppose the analysandum, as for instance when we may say that a circle is a plane curve enclosing the maximum area of a given arc length; that it is the shape that has isoperimetric equality etc. There are numerous such equivalence-theorems, in mathematics and outside it, that would not count as proper analyses. In this respect a certain hierarchy of dependence-relations is often assumed and the requirement is that analysans should be couched in terms that are more basic in the hierarchy than analysandum. I shall not discuss these traditional requirements here and rather focus on further, particularly Fregean principles. I shall present five such principles as forming together a partial basis for a theory of conceptual analysis and as constraining Frege's notion of analytical definition. I begin with stating these principles, indicating relevant aspects of their significance and motivation in his view. I shall then apply them to the more specific notion Frege called 'analytic (*zerlegen*de) definition', which is often the product of the analysis. When it is, it expresses what I call 'analytic explication', which is claimed to be constrained by these principles.

The Principle of implications enrichment – Frege was relatively explicit that the goal of logical analysis of (a sense of) a concept (or of proposition about it), which he compares to chemical analysis of a substance into its elements[5], is that it, and a definition based on it, enable us to prove or expose implications involving the concept, which we approve, but couldn't prove, or didn't even see, before; Hence also its great importance: it not only reveals new implication links, but also brings to the fore the concepts on which the analyzed relies.

This idea of logical analysis finds expression already in the Introduction to *Begriffsschrift* of 1879 (BS) and persists throughout Frege's writings. A clear late expression from his 'Logic in Mathematics' of 1914 (LM) is:

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[5] Frege uses the analogy on various places, e.g. 'Boole's Logical Calculus and My Concept Script' (PW, 36/40), 'On concept and Object' (FR, 182/193), 'Logic in Mathematics' (PW, 208/225), 'Foundation of Geometry II', *Collected Papers* (CP, 302/303).

> For it may not be possible to prove a truth containing a complex constituent so long as that constituent remains unanalyzed; but it may be possible, given an analysis, to prove it from truths in which the elements of the analysis occur (PW 209/226).

I shall call this feature of logical analysis the principle of **implications enrichment**.

It should be noted that this is also important for Frege's notion of consistency (e.g. in his debate with Hilbert), which, for Frege, is a property of a set of thoughts, not of uninterpreted sentences. If these thoughts are not fully analyzed, the risk of inconsistency, he thought, is always there.[6]

This feature of his conception of logical analysis has been widely recognized.[7] It is in fact rooted in a general conception of meaning expounded already in BS, that what he called there 'the conceptual content' (*begriffliche Inhalt*) of a proposition is constituted by its implication relations: Two propositions have the same conceptual content if they have the same implication relations, i.e. if they imply (perhaps with other premises) and are implied by the same propositions. The implication relations are conceived here as constitutive of the content of a proposition. This may also explain the deep philosophical reason for Frege's calling his logical language *Begriffsschrift* – concept script.[8] For, a logical language is designed to conspicuously display the implication relations of a

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6 This has been presented in detail e.g. in P. Blanchette (2007). In FG Frege argues that axioms, as traditionally understood, are (expressions of) true thoughts, and as such their consistency is assured (FG, I, in CP 275/321). He further argues in FG II that Hilbert's axioms are really used as (2nd order) definitions of concepts. Their consistency (and independence) is therefore determined by the consistency (and independence) of the characteristics (*Merkmale*) of these concepts, which are the result of their analysis.

7 Blanchette (2007); Shieh (2008).

8 I expanded on this in Bar-Elli (1985). Some of Frege's early remarks suggest this to be in line with the Leibnizean notion of *lingua characterica*, and he probably took the term *Begriffsschrift* from an article by Adolf Trendelenburg about Leibniz. However, in his late 'Notes for Ludwig Darmstaedter' of 1919 (PW, 253/273) Frege emphasized the difference between his conception and Leibniz's, and even regretted choosing the term for its allusions to Leibniz's conception.

proposition as a function of its inner structure; and to achieve this, the concepts (strictly – concept-senses) of the proposition must be fully analyzed.

Frege often implies, especially in criticizing others' views, that an analysis not satisfying this principle of implication enrichment is pointless, and in fact void (see e.g. PW 221/239 ff.).

It will be claimed, however that this is insufficient, and Frege was much less explicit on further constraints governing such analyses and definitions. These will be our main concern. I shall present some further principles of Frege's philosophy, primarily of his conception of sense (*Sinn*), and suggest that all together they may be seen as forming a basis for a theory of conceptual analysis.[9] These principles are scattered and sometimes quite latent in his writings, but operative both in his critiques of other views and in his constructive development of his own.

The Principle of About – A central thesis of Frege's *Foundation of Arithmetic* of 1884 (FA) is that numerical propositions like 'There are three apples on the table' '**contain an assertion about a concept**' [*eine Aussage von einem begriffe enthalte*, FA §§ 46, 55]. This formulation, unlike many expositions of Frege's view, does not speak of a property of a concept, but is phrased in accord with the context principle (see below) in terms of **propositions**. Moreover, it is phrased in terms of the notion of **about**, which is a propositional notion – a relation between a proposition and things it is about.[10] This, as I argued elsewhere, is a fundamental notion in Frege, posited at the basis of a fundamental Fregean principle, namely that any meaningful proposition is about something(s).[11] Frege

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9 Mind the introductory remarks on this expression.

10 This is missed in some presentations of Frege's view. For instance in his 'Frege's Theory of Number' (reprinted as ch. 6 in his (1983) Parsons presents the thesis by saying: 'Having a certain cardinal number is a property of a concept'. Frege's formulation is not only more cautious in avoiding parson's somewhat unclear 'having a number', but more importantly in being explicitly phrased in terms of proposition and the notion of about.

11 Bar-Elli (1996), ch. 7. I suggested there, against Dummett and many others, to found the realistic basis of Frege's notion of reference on that of about, rather than on the naming relation – a suggestion that seems to me better both philosophically and exegetically. The centrality of the idea of about in Frege occurs already in the

later proposed that the things – whether objects or functions (including concepts and relations) – a proposition is about are the references of its constituent terms,[12] which also implied that a proposition cannot be about something not explicitly referred to by a constituent of it.[13] I shall treat both claims as one thesis and call it '**Frege's principle of about**'.

It is stated by Frege as a basic premise for his construal of quantified propositions, like universal propositions of the form 'All As are Bs', and of numerical propositions like the above, as consisting in an assertion about concepts.[14] In the Introduction to *Basic Laws of Arithmetic* of 1893 (BL) p. 5/ix Frege presents this as his main discovery in FA. It also lies at the basis of his opposition to 'implicit definitions' and of his polemics with Hilbert: One of Frege's main claims there is that such definitions do not determine meaning uniquely, and that '**One surely needs to know what one is talking about**' (*wovon man aussagt*, PW 213/230). This talk of the constituents of a proposition involves substantial views about its logical form, which determines these constituents and on which, Frege thought, vernacular language and traditional grammatical analyses may be profoundly misleading.

Frege's principle of about is often a quite effective constraint on explicating the sense of a proposition when we know what objects (or concepts) the proposition is about, like when we deal with regular empirical objects such as tables and apples, colors and

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early 'Dialogue with Puenjer' (PW, 66-7/59-60) and persists throughout his writings.

12 Cf. Dummett (1981), p. 196 ff. Though Dummett there seems to assume that reference is primary and we explicate the notion of about in terms of it, whereas I suggested in Bar-Elli (1996), ch. 7 the opposite direction.

13 This explicitly appears in SR (58/28). In FA it is explicit regarding objects: '**It is impossible to speak of an object without in some way designating or naming it**' (60), and implicit in the course of the general argument there regarding anything. It is re-affirmed in many of Frege's later writings. For example, in 'Logic in Mathematics' of 1914 he writes: '[Myth and fiction apart] **a proper name must designate something and in a sentence containing a proper name we are making a statement about that which it designates, about its meaning** [*Bedeutung*]' (PW 225/243; see also e.g. 'On Schoenflies' in PW, 180/194-5).

14 Bar-Elli (1996) in particular ch. 7.D.

persons.[15] In such cases knowing the objects – their nature and ways of identification, standard ways of justifying claims about them, etc. – may pose constraints on explicating the thought and the senses (*Sinne*) of the terms concerned, for these senses should express ways in which these things (supposedly known to us) are given. It is for this reason that, say, modeling propositions about tables and apples in arithmetic would not count as explicating their senses or as analytic definitions of them (or of the terms referring to them; more on this later). It is much less effective, however, and in fact quite problematic when we deal, e.g. with abstract objects like numbers. If we don't know what objects the numbers are, we may lack enough constraints on explicating their senses. And this, of course, is the pivotal question of FA: '**When we make a statement of number, what is that of which we assert something**?' [*von wem durch eine Zahlangabe etwas ausgesagt werde*, FA § 45].

The principle of about is also the reason why Frege did not even consider founding the analyticity of geometry on the possibility of modeling it in arithmetic, and ultimately in logic. Frege was of course aware of such possibilities. One might think that they should enable him to turn the reduction of arithmetic to logic into a reduction of geometry to logic (or the logical definitions of arithmetical terms into logical definitions of the geometrical ones), and thus establish the analyticity of geometry. But he never proposed it. The reason for this may be stated in another context in FA: '**But surely, everything geometrical must be given originally by intuition**' (*alles Geometriche muss doch wohl urspruenglich anschaulich sein*, FA 75). It persists in his later writings. In a letter to Hilbert of 27.12.99 (PMC 37) Frege wrote: 'I call axioms propositions that are true but are not proved because **our knowledge of them flows from a source very different from the logical source**, a source that might be called **spatial intuition**'. In talking about sources from which knowledge flows, Frege is not talking of what is involved in the psychological process of coming to know something. He is rather talking of an epistemological issue – the foundations of justifying the truth of this alleged knowledge. Spatial intuition is the source of knowledge of geometry (about which he and Hilbert were talking), and of how the geometrical concepts are given to us. An arithmetical or logical modeling of geometry would be pointless as its analysis, for it would not respect the ways geometrical objects and concepts are given to us by intuition. In other words, it would not respect what geometrical

[15] This has been sometimes called 'identity-dependent' proposition; see e.g. Blackburn (1984).

propositions are about. By the same reasoning the principle explains the pains Frege took in arguing for the logical nature of arithmetic and of numbers in justifying his analysis of them and disqualifying others (see e,g, FA §14). This amounts to a philosophical requirement of an analysis and of analytical definition to the effect that it respects the way (or the standard way) the defined is given to us – its sense. The principle of about is thus connected to Frege's core idea of sense, on which more will be said in the sequel.

I use 'requirement' or 'constraint' rather than 'criterion' or 'necessary condition', and speak rather vaguely of 'respecting' the way the analyzed is given, partly because as noted above, in some cases we may not be clear on what it is that a proposition is about. Yet, even in these 'hard' cases the principle is not void, for we may know enough to know that what the proposition is about is not what a proposed analysis displays (the case of geometry and numbers may serve as an example), and this may be a good reason to disqualify the analysis.

We said that the principle of about may constrain explicating the sense of propositions that deal with objects, like tables and apples, whose nature we know, but that it seems problematic with regards to other objects, like numbers, where we cannot presume such knowledge. Frege's way of overcoming this difficulty, if only partially, was to reverse the order of explanation, namely to explicate the senses of these objects – how they are given to us – by their being constituents of the senses of propositions about them, which are independently explicated, and when this is achieved satisfactorily we may get a grip on what these objects are, or can be. Thus, by explicating the sense of many numerical propositions of the form 'There are n Fs' and of what they are really about, we can get a grip on the meaning of n. This, being the general course of argument of FA, is admittedly put here in a quite general and programmatic way, for it is not at the centre of our concerns, but I trust it that anyone familiar with FA can guess the direction in which a fuller account would proceed.

The Context Principle – Frege understood the notion of about realistically: A proposition cannot be about something (whether an object or a function) that does not exist.[16] Holding this he had to show that any referring term has a reference. Frege in fact

[16] This is a substantial thesis of BL and occurs in many of his writings after that. But it is already clearly stated in his '17 Key Sentences on Logic' (PW 174/189), which is probably an early piece (see Dummett (1981b), pp. 66, 77). In article 10 he

proves this for all referring terms in BL, but in FA he appeals to a broader view relying on a strong version of the **context principle**, according to which if a proposition containing these terms has a clear sense, which determines its truth-conditions, this **suffices for securing the terms references**. Generally, Frege's context principle says that only in the context of a proposition does a term have meaning (FA, Introduction). This can naturally be understood as a general thesis about meaning, namely, that the meaning of a term is its contribution to the meaning of propositions containing it. In FA Frege did not yet have the systematic sense/reference distinction,[17] but in its strong version the principle applies to both and states, as said above, that the meaning (sense) of a proposition secures reference to its terms:

> Only in a proposition do the words really have a meaning [...] It is enough if the proposition taken as a whole has a sense; it is this that confers on its parts also their content. [*Es genuegt, wenn der Satz als Ganzes einen Sinn hat; dadurch enhalten auch seine Theile ihren Inhalt*] (§ 60; cf. § 62).

This was emphasized by Dummett, who construed *Inhalt* here as reference.[18] Many scholars object and proclaim that this goes against Frege's view that a term may have sense but lack reference.[19]

Following e.g. Evans, I believe quite the opposite – that it is this possibility of a referenceless sense that goes against Frege's principles. Although some of Frege's formulations (especially in SR) seem to admit this possibility, they do not, I believe,

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states: 'The sentence 'Leo Sachse is a man' is the expression for a thought only if 'Leo Sachse' designates something (*etwas bezeichnet*)'.

17 But see note 28 below.

18 According to the strong interpretation, the logical formula displaying the analyzed form of e.g. 'there are 3 apples on the table' secures sense to the proposition, thus conferring reference to '3', which doesn't occur in the formula. But there are weaker interpretations in which only the constituents of the formula are conferred reference.

19 For an early such objection see e.g. Parsons ibid, p. 156. See also Dummett (1981), pp. 160-171. Dummett emphasizes that Frege, though admitting it, saw it as a defect in a natural language, and that he thought that semantics cannot allow sentences lacking truth value, but less cautious ascriptions of the view to Frege persist still today.

express his better and considered view. It is rather this possibility which is incoherent with Frege's basic principles, like his adherence to the bivalence principle of classical logic, for, a sentence with a referenceless component is neither true nor false, in defiance of this principle. For a similar reason it doesn't cohere with his view that a thought is essentially true or false. It also hardly makes sense on his view that sense is a way a reference is given to us, since this hardly makes sense when there is no reference. For a similar reason the possibility concerned would hardly make sense on Frege's view that a meaningful proposition is about the references of its terms. All these strongly support the view that a referenceless sense is a problematic, in fact incoherent, idea. And indeed, most of Frege's pronouncements on the topic go in this direction.[20] Whenever a term lacks reference, this shows that it only seems to have a sense, but in fact doesn't. Strictly, **sense for Frege is supervenient on reference**.[21] Yet, he conceived of certain **extensions** here in trying to account for our seeming understanding of expressions which only seem to have sense. These extensions relate, according to Frege, mainly to expressions in mythical or poetical contexts, which he regarded as devoid of sense in the strict sense. When sentences are used to express genuine thoughts (which are their senses), they cannot contain constituents that are devoid of reference, for then they would be neither true nor false. This is a much discussed exegetical issue into which I shall not get further here. These considerations support regarding the context principle and its strong corollary as a governing principle of a logical analysis of the sense of a term. An analysis that does not satisfy it does not show how the sense contributes to the (genuine) senses of propositions containing it, and therefore sins to the very nature of sense and its relation to its reference.

The Core idea of Sense – The sense of a proposition is built up by the senses of its constituents, and these senses express ways in which their references are given to us or

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20 See e.g. 'On Schoenflies' in PW, 180/194-5.

21 In contrast to a widespread (but wrong) presentation of senses as "platonic objects". In addition to the above reasons, the naturalness in which Frege could say that senses of indexicals and demonstratives (like "I", "here", "now", "this", etc) contain elements of the outer context (T, in FR 332/64-5) strongly support this interpretation. Many alternatives (influding dummett's) are not only exegetically, but also philosophically problematic and make the relationship between sense and "its" reference (when it has one) quite mysterious.

presented to us – their *Art des Gegebenseins*.[22] This is Frege's main characterization of sense – its **core idea**. It should be distinguished from another one, made part of the Fregean lore especially by Dummett, construing sense as a 'route to the reference' or a 'way the reference of a term is determined'.

This, I think, has a rather slim basis in Frege.[23] And it differs from Frege's main characterization of sense in its orientation and overtones: Frege's core idea – ways things in the world are given us – is epistemic, directed as it were from the world to us; the latter (Dummett's) is anchored in the philosophy of language, directed as it were from language to the world. The former hardly makes sense for terms lacking reference; the latter can tolerate such an idea more naturally. However, this is not to imply that Frege's notion is detached from the philosophy of language. On the contrary – they are tightly linked. The links, however, are more roundabout and sophisticated than may be suggested by Dummett's formulation and often presented by others. Things in the world are given to us in ways that are constrained and displayed by the conventional meanings of the linguistic expressions that refer to them. By conventional meaning of an expression I mean here mainly the ways it is commonly used by speakers of the language. That Frege regarded it as constraining sense is evident in many of his polemical arguments, even before the systematic terminology of *Sinn*, e.g. FA § 29. Frege's core idea of sense may therefore be briefly put thus: A sense of a term is a way its reference is given to us as this is expressed by the (linguistic) conventional meaning of the term referring to it.

This appeal to language as constituting ways things in the world are given to us marks not only the objective character of Frege's notion of sense and the exclusion from it

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22 This is the way it is introduced in SR and used widely later. I shall not be strict here in distinguishing sense of a term from sense of its reference and trust it that the context makes clear which is meant. Frege's 'core idea' makes ascribing sense to a reference quite natural, and he often speaks of a sense as attached to a reference (e.g. in SR 27, FR 153). Likewise in talking of definitions I shall use 'the defined' for a term and for its sense.

23 A quite common (not Dummett's) explication of this 'way of determining', as a condition whose (sole) satisfier is the reference, seems flatly wrong, as it suggests a predicative construal of the sense/reference relationship to which Frege explicitly opposed. See e.g. his letter to Husserl of May 1891, PMC 61-3; SR, 64/34-35 in TPF (158 in FR); PW 194/211.

of subjective psychological factors, but also his divergence from Kant and his difference from e.g. Husserl. It is one of his main revolutions in philosophy. But this appeal to language should not undermine the general epistemic orientation of Frege's notion of sense as expressing a way something in the world is given to us. Explicating the sense of a term should therefore respect this way its reference is given us.[24] It is for this reason, as noted above in discussing the principle of about, that Frege was not satisfied with the mere possibility of modeling arithmetic in logic, but devoted much effort to arguing that numbers, as they are used in numerical propositions and in arithmetic, are logical objects and are given to us as such.

The Justificatory Significance of Sense – This notion of sense, encapsulating ways in which things in the world are given to us, has **justificatory significance**, particularly in forming an epistemic, non-deductive justification of basic truths (axioms) which cannot be derived from other more elementary truths. I have defended elsewhere its ascription to Frege and argued that it is crucial for understanding his notions of analyticity and apriority, particularly in applying them to basic truths, for these notions, he insists, concern only justification (Bar-Elli, 2010).[25] If accounting for the analyticity of arithmetical propositions and for the apriority of geometrical ones is considered vital for their analysis (as Frege, I believe thought), the justificatory role of sense becomes predominant. I shall not repeat the details here, but add another aspect of it that has not been given, I believe, its due attention.

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[24] In BL and later writings Frege often talks of sense as a constituent of a thought (which is the sense of a proposition). This, though having the merit of being in accord with the context principle, can be interpreted in various ways and can hardly settle the issue between the two characterizations discussed in the text. This and its relation to the core idea of sense is discussed in Bar-Elli (2001)

[25] To the references given there for Frege's thinking of non-derivational justification (of axioms) one could add the above letter to Hilbert of 27.12.1899 (PMC 37) in which Frege writes: 'I call axioms propositions that are true but are not proved because **our knowledge of them flows from a source very different from the logical source**, a source that might be called spatial intuition'. Spatial intuition is mentioned here because they were talking about geometry, but the point I believe is general, and it shows that knowledge of axioms have a 'source' from which it 'flows'; appeal to this source and the way knowledge of the axioms flows from it is a form of non-derivational justification of the axioms and their knowledge.

In his extensive attack on formalistic approaches in *Grundgesetze der Arithmetik* vol. II Frege contrasts what he calls 'meaningful arithmetic' with 'formal arithmetic'. In attacking the latter, Frege's main arguments (and worries) concern the possibility of accounting for the **applicability** and the **justification** of a system of rules and axioms− arithmetic in that case. These he argues must be couched in terms of senses and references and are therefore not available to formalists.

He argues at length that a formalistic approach cannot account for the applicability of arithmetic:

> Arithmetic with no thought as its content will also be without possibility of application (*Anwendung*) [...] Why can arithmetical equations be applied? Only because they express thoughts [...] It is applicability alone which elevates arithmetic from a game to the rank of a science (Gg II §91; TPF 187).

This applicability he emphasizes 'cannot be an accident', and calls for grounding and account, which the formalist cannot provide (ibid. §89). In § 92 there he argues that since arithmetic and numbers have applications in various domains, their applicability is an internal arithmetic issue, and '**to this end it is necessary above all that the arithmetician attach a sense to his formulas**'.

However, it is not only the universal applicability of arithmetic which necessitates appeal to sense and reference; it is also the need to **justify** its base – rules and axioms: '**It is necessary that formulas express a sense and that the rules be grounded** (*Begruendung finden*) **in the reference of the signs**'. (ibid. §92 p. 188; cf. § 94). In §101 he continues to argue that in meaningful arithmetic 'We start from what is known about integers, **from connexions grounded in their nature** (*in deren Wesen begruendet sind*)'. It is evident therefore that for Frege rules (and axioms) of arithmetic (and any genuine theory) are grounded and justified by the senses of its constituent-signs (which, he claims, are unavailable to the formalist). The justificatory role of sense is in effect not only with regard to axioms (whether analytic or a priori) or rules, but also with many regular aposteriori propositions, like some perceptual ones. In many such cases truths are not justified by deriving them from other, more basic ones, but by the ways their objects are given us – their senses. Analysis of these senses should therefore enable to expose this justificatory role and explain it.

We have briefly surveyed some philosophical doctrines, which, even if Frege, as some would claim, did not express as clearly as one could wish, are fundamental in his philosophy. Chief among them are: 1) the principle of implications enrichment by logical analysis; 2) the principle of about, including its corollary that understanding a proposition involves knowing what it is about; 3) the strong context principle, including its corollary that explicating the sense of a proposition secures reference to its constituents; 4) the core idea of sense and its corollary that explicating the sense of a term should answer to the ways its reference is given to us; and 5) the justificatory role of sense – the view that the sense of a term – a way its reference is given to us – may have a justificatory role, particularly in justifying basic truths containing it, which cannot be derived from other, more elementary ones.

These principles are the basis of much of Frege's criticism of other approaches. The principle of implication enrichment is often appealed to in his criticizing various analyses and definitions based on them, which do not show up in proofs and do not reveal any implication relation, and therefore are either senseless or of no use (see for instance, his long critiques of Weierstrass in 'Logic in Mathematics' 221/239 ff.). The principle of about is basic, for instance, in his criticism and rejection of formalistic approaches. This and the context principle are basis of many of his polemics against empirical and psychological approaches. The justificatory role of sense is, I believe, important in understanding his departure from Kant on the notions of analytic and a priori.[26] But, perhaps even more importantly, these principles together with the core idea of sense are basic in the constructive development of his own argument in FA.

Clearly, each of these principles deserves further clarification and elaboration, which partially I have tried to carry out elsewhere. The point I wish to suggest here is that this cluster of ideas forms a basis for a theory of logical analysis in the sense that they are principles such an analysis should respect. As remarked above, I use this rather vague expression rather than talk of them as criteria or necessary conditions mainly because on some cases their applicability may be deemed unclear: It may be unclear, with respect to the principle of implication enrichment, whether e.g. some implications are ones we approve; It may be unclear, with respect to the principle of about, what some propositions are about; It may be unclear what is the way something is given to us as expressed by a term referring to it, etc. However, here as elsewhere, the fact that a principle is not always

[26] I elaborated on this in Bar-Elli (2010) and (2014).

applicable does not mean that it is always inapplicable. We have given above many Fregean examples of their applicability, and will give some more in the sequel in discussing them as philosophical constraints on analytical definitions, to which I now turn.

Analytical Definitions

Definitions are of course crucial for Frege's project, and he was well aware of that to the point he could write that arithmetic is derivable from definitions by means of logical laws (rather than from logic by means of definitions; see e.g. his letter to Liebmann, PMC 92-94). He repeatedly criticized definitions given by other mathematicians and logicians, and posed some stringent conditions for proper definitions (FA §§94-103; Frege expands on it in *Grundgesetze*, vol II §§55-164, tr. in part in FR 258-290 and in TPF). To mention some: a definition should be **conservative** – logically, the defined is eliminable and no thought should be provable from it that is not provable without it, although psychologically it may be indispensible, and proofs and thinking without it would be intolerably cumbersome;[27] It should be **complete** – Frege objected to partial or piecemeal definitions determining meaning only in some contexts, leaving open the meaning of the term in others;[28] This includes the demand that a definition of a concept should be 'sharp' – it should determine for any object whether or not the concept applies to it; It should determine the meaning of the defined **uniquely**; It should be given in terms of ***Merkmale*** – concepts of the same level as the defined, keeping strictly the distinction between first and second level predicates.

The ultimate reason for these requirements is to prevent or minimize risk of inconsistencies. Actual non-appearance of contradiction cannot suffice here: '**Rigour of a**

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[27] In explaining that in thinking and talking we are often unaware of the sense of our words, Frege writes: 'So if from a logical point of view definitions are at bottom quite inessential, they are nevertheless of great importance for thinking as this actually takes place in human beings' ('Logic in Mathematics', PW, 209/226).

[28] These requirements are well known and I shall not expand on them. For this and the change in Frege's view on 'piecemeal' or 'contextual definitions', from FA (in which he thought that such definitions always depend on some justification for the existence and uniqueness assumptions) to BL (in which he thought that such justifications are in general impractical), and for the bearing of the issue on the status of Hume's principle in his system, see Shieh (2008), § 3.

proof remains an illusion, however complete the chain of inferences may be, if the definitions are only justified retrospectively by the non-appearance of any contradiction' (FA, Introduction IX; cf. also FA §94, and BL vol. II, §55).

But Frege was not only well aware of the importance of formal requirements of definitions, but also of their philosophically problematic nature and of philosophical requirements they should satisfy. Already in the Introduction to FA he writes:

> To those who feel inclined to criticize my definitions as unnatural, I would suggest that the point here is not whether they are natural but whether they **go to the root of the matter** [*den Kern der Sache treffend*] and are logically beyond criticism (FA, p. xi).

'Natural' probably means here that anyone would recognize the definition as giving the sense of what is defined, or even as offering a synonym. As if anticipating later objections, Frege is well aware here that his definitions are not statements of synonymy or of an even weaker meaning relation that would make them 'natural'.[29] He is also claiming that satisfying the formal-logical requirements, being 'logically beyond criticism', is perhaps necessary but not sufficient. Definitions should also 'go to the root of the matter'. What does that mean if synonymy and naturalness are too strong, modeling and even sameness of extension[30] – too weak, and being logically beyond criticism – insufficient?

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[29] The claim was raised by Husserl shortly after the publication of FA. Parsons cites Follesdal as claiming that in his *Aufsaeze and Rezensionen*, 11-12 Husserl developed a distinction parallel to Frege's sense/reference one independently of and prior to Frege (see Follesdal's article cited by Parsons in *From Kant to Husserl*, Harvard, 2010, n. 22 p.145). Let me remark in this connection that although Frege's systematic *Sinn/Bedeutung* distinction appears in his papers of 1891-1892, the basic idea of the distinction, as I argued in Bar-Elli (2006), appears already in § 8 of *Begriffsschrift* of 1879 (confined there to identity statements), and in § 62 of FA, both of which we know Husserl to have read shortly after their publication. So it is still possible that Husserl's distinction was inspired or influenced by his reading these Fregean sources before 1890.

[30] In his review of Husserl Frege notoriously seems to adopt an extensionalist view. This has been often criticized as plainly insufficient. See e.g. Dummett (1981), 148-54; Shieh (2008).

Frege often distinguishes between definitions within a constructed system and explanations of the senses of terms, which are external and preparatory to the system, and serve, especially for primitive terms, in smoothing understanding and communication. He calls these explanations *Erlauterungen* (translated in FG II, 'elucidations'). But something akin is often encountered also within a system, when we deal with definitions of familiar terms with a long standing use. And this is the case with most definitions in Frege's own systems.

In his posthumous 'Logic in Mathematics', Frege distinguished 'constructive' (*aufbauende*) definitions from 'analytical' (*zerlegende*) definitions[31]. The former are in fact stipulations for shortenings of complicated expressions. They are, he says, like '**receptacle for the sense… we conceal a very complex sense as in a receptacle – we also need definitions so that we can cram this sense into the receptacle and also take it out again**' (PW 209/226; cf. FG in CP 274/320). The latter, analytical definitions, are definitions of terms in use, whose meanings are at least partially known and need in some way to be maintained and clarified by the definition. These are of course the focus of our (and his) interests, for they are the results of what he calls 'logical analysis'. For them the distinction between the *propaedeutic* analysis and the definition of a term in use within the system is somewhat blurred, for the former is evidently the rationale for the latter.

Frege then distinguishes two main sorts of analytical definitions. In the first, the agreement between the complex sense of the definition and the sense of the '**long established simple sign […] can only be recognized by an immediate insight**' [*unmittelbaren Einleuchten*]. In such a case, he says, the definition should be better termed an axiom. Frege seems to assume here that if the senses of A and B are clearly recognized one cannot be in doubt on whether or not they agree (ibid. 211/228). But what of the other cases where such a doubt can arise?

Frege's answer divides into two parts, one of which is in fact what has come to be known, following Carnap, as '**explication**'.[32] This amounts to a suggestion to replace a

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31 The English 'Analytical' here should not be confused with his notion of analyticity or analytical judgment (see Bar-Elli 2010).

32 This is distinct, though related to Frege's use of *Erlaeuterung* (translated as 'explication' in FG II, 300/301, which is closer to Carnap's 'explanation of the explicandum'. Carnap attended Frege's course 'Logic in Mathematics' in 1914 (see his 'Intellectual Autobiography' in Schilpp, ed. (1963). The article in PW consists of

term in use, whose meaning may be obscure, vague, only dimly or partially grasped, by a new term, precise and well defined within a theory. For such replacement to count as explication or analytic definition there should be a significant overlap in the use conditions of the two terms, which will include the paradigmatic and clear-cut cases of the former. This may be considered an extensional condition, and the whole idea, as Carnap emphasized, a **pragmatic suggestion** for replacing, for specific purposes, a term in use by a new precisely defined one. This has no pretension of being correct or of the latter providing logical analysis of the former, but only of being pragmatically expedient. To this effect Frege also talks of *Erlaeuterungen* here, which was translated in PW as 'illustrative examples'. Whether or not the translation is a happy one, this seems to be only one kind of *Erlaeuterung*. Another kind, sometimes translated as 'elucidation', is e.g. '**sentences containing an empty sign, the other constituents of which are known**', which, as Frege says, may '**provide a clue to what is to be understood by the sign**', PW, 214/232).

However, in the second part of his answer Frege goes beyond this and writes:

> If this [whether the senses agree] is open to question although we can clearly recognize the sense of the complex expression from the way it is put together, then the reason must lie in the fact that we do not have a clear grasp of the sense of the simple sign, but that its outlines are confused as if we saw it through a mist. **The effect of the logical analysis of which we spoke will then be precisely this – to articulate the sense clearly** [*Das Ergebnis jener logischen Zerlegung wird dann eben sein, dass der Sinn deutlich herausgearbeitet worden sein*]. Work of this kind is very useful (ibid. 211/228).

This passage is of great importance, both in itself and in understanding Frege's attitude towards his own definitions in FA (as well as in BS and BL which are almost all of that kind). First, it says that we may question whether two senses agree even when in fact they do, which implies that our conviction that A and B differ in sense may be mistaken – it may arise even when in fact their senses agree.[33] This is rooted in Frege's view, clearly

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the notes for this course, so there can be little doubt that Carnap got the idea of explication in ch. 2 of his *Foundations of Probablity* from Frege.

[33] This is a feature of the objectivity of sense that is often ignored. It is an important reservation to a widely accepted doctrine of the transparency of sense, namely that if two senses agree, one who understands them cannot fail to realize that. This can

expressed in this passage, that a sense can be only partially or imperfectly grasped, and this is crucial for understanding his notion of logical analysis. For, a successful analysis articulates clearly a sense that was unclearly grasped (or that stands to it in particular relations).[34] Secondly, Frege's ambitions seem here obviously much higher than Carnap's. For, he implies, what Carnap tried to avoid, that an analytical definition is the result of conceptual analysis of the old term (i.e. of its sense), where the question of the correctness of this analysis is pertinent. Such definitions, as Frege says, may give rise to judgements that 'extend our knowledge' and are in this sense synthetic (in Kant's though not in Frege's sense); and yet they are contained in the definitons 'as plants are contained in their seeds' (FA §88)[35] To distinguish it from Carnap's explication, I shall therefore call it '**analytical explication**'.

But unfortunately Frege stops there and does not answer his initial question, except for the self-evident case, and does not say how we can judge the correctness of an analytical definition, and to what such a judgment amounts. We can thus say that Frege here answers one aspect of the 'paradox of analysis'[36] and explains that analysis can be important and informative in giving a clear and clearly recognized expression for an

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at most be said of one who understands them fully. But this is a very problematic notion.

34 The point was elaborated by Tyler Burge, e.g. in his (1990), though Burge argued that a successful analysis need not preserve the sense of the analysandum. This was recently challenged e.g. by Nelson (2008), who argued that at least since 1914 Frege's notion of analysis did require such sense-preserving. What can change, according to Nelson is not sense but what he calls 'conception'. I shall not go here into the cogency of the suggestion and its point, for as Nelson himself admits it is non-Fregean and in fact goes against the sense/reference distinction.

35 I elaborated on this conception and its relationships to Kant's in Bar-Elli (2014).

36 See his Review of Husserl's *Philosophie der Arithmetik* (FR 224), in which, following Husserl, he gives a clear formulation of what has come to be known as the paradox of analysis. Notoriously, as Dummett pointed out already in ch. 2 of his (1991a) and many had further explained (e.g. Shieh (2008), he doesn't answer it there, and the view he presented there clashes with both his view in FA and his later views, e.g. in LM.

unclear and unclearly recognized concept[37]. He seems to presume here that such an analysis is meaningful and expresses an objective thought. But he doesn't state what this thought is and what the constraints on its correctness are.

Some form of the above 'extensional condition' is obviously necessary, so that analysans should meet at least paradigmatic cases of the applications of the analysandum. But Frege, as suggested above, aimed higher. As the term 'analytical definition' suggests and as Frege makes explicit in the above passage, it amounts to a logical (conceptual) analysis, and as we saw before it should 'go to the root of the matter'. Frege poses the question of how we can know, or what it means that such an analysis and the definition resulting from it are correct. But he doesn't really answer it except for the case that it is immediately evident. If the problem was not extremely difficult and one of philosophy's perennial ones, this would be most surprising and almost inexcusable. For, as stated above, almost all of Frege's important definitions are of that kind and beset with this problem. Without at least some hints at the answer we might feel at a loss in understanding the heart of his enterprise – what he thought he was doing in his logical analyses.

The problem of the correctness of analysis (and the related 'paradox of analysis') in Frege were recognized already by Dummett (e.g. 1991 (see pp. 30-31, 143; see also his 1991a ch. 2), where he states that Frege did not have a satisfactory answer to it. Dummett suggests that an analytic definition ought 'to come as close as possible to capturing the existing sense' (i.e. of the analyzed; 1991, 152). But, Dummett's added explanations notwithstanding, 'as close as possible' is too vague and 'capturing the existing sense' is unclear – in fact explicating it is our main problem.

Two more recent papers dealing in detail with the subject are Blanchette (2007) and Shieh (2008).

Blanchette rightly emphasizes that the Frege-Hilbert controversy depends on realizing that for Frege implication and consistency concern thoughts and contents (fully analyzed) while for Hilbert they concern formal sentences. She explains in detail that for Frege the consistency of a theory is no guaranty of the consistency of its models (where a model of a theory shares its logical form while re-interpreting the non-logical terms), whereas for Hilbert the passage is unproblematic The reason is that for Frege consistency

[37] Mind the introductory remarks on the problematic nature of analyzing a concept, for strictly, analysis is primarily of sense rather than of reference (concept, function or object).

and implication concern thoughts, and therefore depend on their full analysis and meaning. Thus the consistency of a theory T^g is no guaranty of that of T^r, for this depends on the full meaning of the (non-logical) terms of T^r (see end of FG I, in CP 375/284).

She talks of analysis of propositions, and says: 'Frege takes it that the two propositions [the analyzed and the analyzing] are sufficiently similar that they share logical grounds and entailments', and later she says more cautiously: 'thoughts expressed by ***analysandum***-sentence and ***analysans***-sentence are sufficiently similar that the logical grounds and implications of one are a sure guide to those of the other. This is the assumption that underlies all of Frege's appeals to conceptual analysis in the course of his foundational investigations.' It is not entirely clear what 'a sure guide' is, and I shall not expand here on the problems applying the notion of analysis to propositions rather than their constituents. But my main worry is that Blanchette seems to focus on what we called the principle of implication enrichment, and what is lacking in her account is appreciation of the other principles we discussed and their role in forming philosophical constraints on conceptual analysis and analytical definitions. It is far from clear that sharing grounds and implications capture all of them; in fact, it is quite clear that it doesn't.

Shieh (2008) traces the tension between Frege's views on definition in FA and LM.[38] There are some minor points I find faulty in his presentation. One, which applies also to Blanchette, is that Shieh (along with many scholars) talks (e.g. in § 4) of 'the structure of a concept'. If 'concept' is used in Frege's mature sense, this, as noticed above, is objectionable − a concept in this sense, does not have a structure (cannot be complex), only its sense can (as noted above, derivatively this can be ascribed to the concept when conceived under the sense). Another is that throughout his discussion Shieh appeals to the 'sense-identity' requirement, i.e. that in an analytic definition the senses of the definiens and definiendum should be identical. Though the point is disputable, I think that senses for Frege are not objects,[39] and therefore talking of identity here is problematic anyhow, and particularly with regard to analytic definitions. Shieh identifies a thought with the truth-conditions of a sentence, which Frege never didfor Frege a though t expressed by a sentence determines its truth-conditions but is a far richer notion, expressing a particular way in which these truth-conditions are conceived.

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38 This was described and discussed in detail already in Dummett's 'Frege and the Paradox of Analysis', ch. 2 in his (1991a).

39 I have elaborated on this in Bar-Elli (2015).

But even waiving these worries, I think that Shieh's sharp dichotomy between senses being either identical or completely disjoint ignores intermediate possibilities, like Dummett's 'analytic equivalence' (1991a ch. 2) and the possibility presented above the explicatory relation between senses: Senses that do not fully agree can still stand in this explicatory relation, satisfying our above conditions, and thus be not fully disjoint.

Following the above sharp dichotomy, in discussing the problem of analytic definitions in LM Shieh concludes that in the new system, based on strict definitions 'they or their senses **have nothing to do** with arithmetic as we have used it previously' (end of §5, my emphasis). He also says that according to Frege's late view (e.g. in LM) 'the proofs of *Grundgesetze* are of mathematical statements **unrelated in sense** to existing arithmetical statements'. This seems implausible and foreign to the main tenor of Frege's view.

We have seen before that other scholars (e.g. Nelson 2008) also assume that Frege's notion of analysis requires that the senses of analysans and analysandum be the same. I cannot get here into a detailed discussion of the issue, but will only remark that Frege doesn't explicitly say, even in LM, that a successful analysis always results in their having the same sense. His formulations (e.g. his talking in the above quote of whether senses 'agree') allow for a more flexible view of the relationships between analysans and analysandum, in which their senses correspond, agree etc., but are not strictly the same. When an analysis terminates in a definition, and this is incorporated into a system, then one can say that the senses are the same, simply because the one (definiens) fixes the other (definiendum). But this is not the case with which we are concerned here. By the main tenor of our discussion it seems that in a significant analysis, sense according to Frege, is not strictly preserved. In any case, whether we say, as I think we should, that analysis may result in some change of sense, or with Nelson that the senses must be the same, while what changes is one's 'conceptions',[40] we still face the problem of the correctness of the analysis, which these constraints are supposed to govern. A successful analysis, I surmise, results in a clear articulation of a sense that determines the same concept as the original

[40] Relegating the change to something like Nelson's finer grained 'conceptions', even if spelled out clearly, would run the risk of inserting a wedge between sense grasped, and understanding a linguistic expression, which is constituted by these finer-grained notions. This may be quite a high price to pay.

one, and stands to the original sense in the explicatory relation governed by the principles we spelled out above.

Shieh's positive suggestion amounts to an epistemic twist of the original problem. He briefly mentions Burge's distinction between expressing a sense and grasping the sense expressed, and claims that the former is independent of the latter. He relates this to the difference between a full and clear grasp of a sense and a partial and unclear one, which he construes in terms of the ability to derive the rules of use (of a term) from 'a grasp of the rules governing the use of it' (next to the last paragraph). One could perhaps think that the apparent circularity here is due to a slip at the end of the above quote, which should perhaps be '...governing the use of its defining terms'. But this charitable reading is doubtful, for Shieh in fact repeats the circularity by construing the grasp of a sense so that 'we would have to derive our [rule-governed] uses of those expressions from our knowledge of their senses'. This in itself could be granted, but in the subsequent sentence he implies that knowing the sense of an expression is grasping 'the rules governing its use' (§ 6). This again seems to render the view hopelessly circular. But even though the above difference between a partial and full grasp is indeed Fregean, it is doubtful whether its construal in terms of 'rules of use' and their derivation is. A detailed argument for this will take us too far afield, and Shieh's remarks on that are too brief to allow a detailed examination.

In general, what I believe makes these explications of Frege's notion of analysis insufficient is a lack of full cognizance of the philosophical constraints on explicating sense in Frege's theory. Evidently, I don't pretend to have presented a full answer to the question of analysis. But I would propose that our above principles suggest philosophical constraints and conditions for achieving this: Analytical definition, analytical explication and logical (conceptual) analysis in general should expose and enable to prove implication relations we intuitively wish to hold concerning the analysandum (implication enrichment principle); they should retain and clarify what propositions including the terms concerned are understood to be about (principle of about); they should articulate the sense of these propositions in a way that secures their constituents reference in accordance with the strong context principle; they should explicate this sense in a way that is responsive to the way its reference is given us (the core idea of sense); and they should articulate the senses concerned – ways their references are given to us – in a way that forms a basis for justifying basic truths and axioms about these references.

Philosophical constraints on definitions has been a major concern in philosophy at least since Plato and Aristotle. It was a major topic in the debate between realism and nominalism and between extensionalists and essentialists throughout the ages. Frege had many reservations about classical forms of definition and did not participate in these debates in these terms. But his views about analytical definitions, his polemics against many other approaches and his own constructive way in FA and other writings, suggest a new path in this celebrated history, and our five principles are the basis of it.

Now, as stated above, although Frege doesn't say so explicitly, most of his own definitions e.g. in FA, are what we called analytical explications. And the above principles are implicit in the course of argument of FA. The extensional condition is met by the proof that the definitions concerned offer a logical model of the basic truths of arithmetic (the so called 'Peano axioms'). The principle of about is basic in Frege's view that quantified statements are about concepts, and in the rationale of his doctrine of logical objects, for, logical truths, like any proposition, must be about some things. It is also appealed to in regarding the universal generality of arithmetic as a reason for regarding it as part of logic[41], and this, as we have seen, was also the basis of Frege's account for the universal applicability of arithmetic. Accordingly, the definitions also explicate and clarify what the arithmetical propositions are 'really' about – namely, logical objects and functions defined over them. They explicate the senses of the arithmetical propositions in terms of logical notions whose senses are beyond doubt, thus securing the arithmetical terms references, which render the propositions containing them objectively true or false, in accordance with the strong context principle. These definitions also reveal how the arithmetical objects are given to us – namely, as logical objects given by basic features of our ability to think. They thus form a basis for epistemic, non-deductive justification of basic truths couched in their terms, which is required for regarding them as analytic. A parallel argument works e.g. for the apriority of geometry, but we shall not follow it here.

We may thus say in conclusion that where B is a successful logical (conceptual) analysis of A their senses determine the same reference (concept), while B clearly articulates a sense that stands to that of A in the particular explicatory relation constrained by our five principles. These principles, as noted above, may fall short of being necessary conditions applicable to any proposed analysis, but they are always pertinent in assessing it, and they may disqualify some analyses and motivate and support others.

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41 See also e.g. FA §14.

As Frege makes clear on several occasions his main concerns in FA, and since the *Nachwort* for *Grundgesetze*, his concerns with extensions and the problem his basic law V encounters, were ultimately rooted in these philosophical aspects of his explicatory definitions.[42] For, alternative courses of bypassing these problems (e.g. in axiomatic set theory) would seem to him of limited philosophical significance if they did not form a basis for analytical explications of the nature of arithmetical and logical objects (i.e. extensions) and of how they are given to us. Frege states something like this in the concluding sentences of his appendix (*Nachwort*) to the second volume of Gg (1903). After discussing the problem raised by his axiom V and some possibilities of replacing it he says:

> The prime problem of arithmetic may be taken to be the problem: How do we apprehend logical objects, in particular numbers? what justifies us in recognizing numbers as objects?

Frege then expresses his current belief that the question has not been finally solved, but that his efforts were on the right track. The ultimate reason for this is the philosophical principles of an analytical explication that shaped this track, some of which were proposed above. And although I have talked of these principles separately, I hope it is clear that there are deep inter-connections between them, to the point we can talk of them together as constituting a skeleton of a theory of logical (conceptual) analysis.[43]

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42 Frege had qualms about extensions prior to Russell's Paradox, and admitted his axiom V to be not as evident as other logical axioms. But after 1903 he gradually became more and more skeptical about the appeal to extensions, and the reduction of arithmetic to pure logic. This is a fairly well known story I shall not repeat here. See a review of this e.g. in Parson's ch. 5 of *From Kant to Husserl*, Harvard 2010.

43 Most of the above Fregean Principles were raised by him in the course of his logistic project that we would now perhaps call reducing arithmetic to logic. Though there are many kinds of reduction in many areas, I believe (without being able to argue for it here) that for many of them, particularly in 'open' domains like e.g. the mental, in order to carry the philosophical significance many people attach to them, they should satisfy the conditions of analytical explication sketched above.

Chapter 10: A Fregean Look at Kripke's Modal Notion of Meaning

In *Naming and Necessity*[1] Kripke criticizes Frege for conflating two notions of meaning (or sense),[2] one is meaning proper, the other is the way reference is determined (p. 59). More precisely, Kripke argues that Frege conflated the question of how the meaning of a word is given or determined with the question of how its reference is determined. The criterial mark of meaning-determination, according to Kripke, is a statement of synonymy: if we give the sense of "a" by means of "b", then the two expressions must be synonymous. The criterial mark of reference-determination is knowledge, typically a priori, of the truth of their identity: if the reference of "a" is given by "b", then we know a priori that a is b. Kripke then argues that Frege's conceptions both of meaning-determination and of reference-determination were wrong, and proposes an alternative picture of reference-determination.

I shall discuss some details of Kripke's arguments in the last part of this paper. In the meantime I wish to point out, in very general terms, that the main flaw Kripke finds in Frege's conception is that it is what we may call "over-cognitive": Frege, according to Kripke, mistakenly construes the reference of a term as being determined by beliefs or conditions allegedly known to the speaker. Frege then, according to Kripke, compounds mistakes by identifying this with the meaning or sense of the term. Much of this attack on the Fregean cognitive conception of meaning is couched in terms of a detailed critique of a descriptive theory of the sense of (proper) names, which is the ostensive target of most of Kripke's arguments. The tacit assumption is evidently that a cognitive conception is manifested in a descriptive theory (the sense of) of names.

The relationships between a cognitive conception of sense and the descriptive theory of names are rather elusive. Many theorists assume (with Kripke) that the latter is implied by, or part of the former. They also assume that this was Frege's view. NN can be read rather narrowly as launching an attack on the descriptive theory of names, which Kripke ascribes to Frege. This, of course, is important enough, independently of

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1 S. Kripke, *Naming and Necessity*, Harvard University Press, 1980 (1972), hereafter abbreviated as NN. Page numbers refer to this book unless otherwise stated. For other abbreviations see list of references.

2 "Meaning" has often been used as a translation of Frege's "Bedeutung". Kripke sometimes uses it as akin to Frege's "Sinn". I shall do the same.

whether the ascription of the descriptive view to Frege is correct. (I shall come back to it in the last part of the paper.) On this reading Kripke's arguments against the descriptive conception of names are, I believe, quite compelling. However, NN can be read more ambitiously as an attack on any cognitivist conception of meaning and of reference-determination. It is, I believe, on this more ambitious reading that NN has been justly regarded as one of the most important landmarks in the philosophy of language, as cutting deep into the basic conceptions in terms of which we construe the notions of meaning and reference-determination. It is mainly with the ambitious reading that I shall be concerned here. On this reading Frege's actual view deserves careful examination, for it, or a particular interpretation of it, is widely conceived as one of the most important versions of a cognitive conception of meaning. However, as will be spelled out later, I believe it is, or at least suggests, an important version of a cognitive conception of meaning, which is not committed to the descriptive theory Kripke attacks.

The alternative Kripke proposes for reference-determination is couched, accordingly, in terms of two non-cognitive factors: (a) causal relations between speakers and between them and objects or situations in the real world, and (b) intentions of co-referring--intentions on the part of a speaker to retain the reference of previous links in the causal chain by which the name has passed on to him. Reference, according to Kripke, is determined not by any piece of knowledge or by beliefs available to the speaker, but by a certain kind of semantical intention[3] and by causal

[3] The exact nature of this intention raises some problems. One of them, pertinent to our present concerns, is whether this intention is constrained by epistemic considerations--whether one can form such an intention, on a specific occasion, contrary to his firm existential beliefs. One may well argue that given certain beliefs and epistemic conditions, the appropriate intention may be unavailable to the speaker, and that his use of a name will thus become spurious. I heard from a Kabbalist long stories about Bilar–the king of devils. Since I don't believe there are such creatures, it seems to me that I cannot form the appropriate intention (to refer to whatever the Kabbalist was referring to), even if, contrary to my convictions, it turns out that there are devils and they have a king, and the Kabbalist did refer to something.

factors in the actual situations in which a speaker utters the term and by which he has mastered its use.

In this wide conception of NN, it is one of the main origins of the externalist position, according to which the meanings of various linguistic expressions, such as proper names and terms for natural kinds, are essentially related to things in the actual world, without which they cannot be grasped. This view, whose hold in philosophy of language and philosophy of mind strengthened in recent generations, is not opposed, I believe, to Frege's view. On the contrary, it is in full harmony with Frege's view that the sense of a term is a way its reference is given to us, which forms a particular supervenience relation between sense and reference. From this perspective Frege should be regarded also as one of the founding originators of the externalist position.[4]

One may wonder whether this causal theory of reference-determination has not gone too far in its non-cognitive orientation. For the notions of reference and reference-determination are to serve in a theory of meaning whose ultimate goal is to account for our use of words in our language. The causal chain, of which the speaker need not be, and usually is not, aware, may lead to a referent he doesn't know, and of which he hasn't even dreamt; it may lead to something utterly and categorically incompatible with whatever he intended to say. In fact it may lead to nothing at all. This may be entirely unknown to the speaker, and hence hardly affect his use of the terms involved. If the picture of reference-determination is thus detached from anything that is within the cognitive horizons of the speaker, how can it provide for an account of his use of these words?[5] For this reason, it seems to me particularly desirable to construe a version of Kripke's causal chain theory as characterizing one mode in which a reference is given to us, within the bounds of a cognitive conception of sense. I shall not go here into details, but rather remain at a general level, arguing that the alleged contrast between this theory and Frege's cognitive conception are over-stated.

A quite common retort here (possibly held by Kripke himself) is that semantics and theories of reference should be separated from the cognitive aspects of our use of language. These aspects are of course important but they should be dealt with in other

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4 I have expanded on this in my book (Bar-Elli (2009), chs. 3,7,8,10.

5 Related points have been widely discussed in the literature. See, for instance, Dummett (1973), pp. 146-51, and (1981 182-86. I shall not go into it here.

terms and by separate theories.[6] I believe that such a move undermines one of Frege's greatest insights: that a theory of thought, in its wide sense, cannot be thus detached from a theory of meaning; it is dependent on it and constrained by it. Again, the issues involved are very wide and intricate, and I shall not go into them here.

In the first part of what follows I shall describe what may be called Kripke's modal conception of meaning, which I regard to be a novel and important notion of meaning. This, along with the non-cognitive attitude towards meaning and reference-determination sketched above, form, perhaps, the most significant contribution to the conception of meaning in this celebrated monograph. It may appear that the modal notion of meaning is a feature of the non-cognitive approach to meaning. The relationship between these two central doctrines is, however, more complicated than might at first sight appear, and I suggest that the modal conception is not, in itself, opposed to the cognitive conception, and can be incorporated in it. In the last part of this paper I shall examine Kripke's arguments against Frege's (cognitive) conception of sense from a Fregean perspective. I shall try to show that the modal conception of meaning in itself does not suffice for discarding Frege's cognitive conception of sense and of reference-determination. A Fregean cognitive conception of meaning may be, therefore, compatible with something like the modal conception of meaning--or so I shall argue.

I mean to focus here on NN exclusively, but I cannot avoid mentioning an interesting link between this non-cognitive trend in NN and the view Kripke suggests, in Wittgenstein's name, in his later book on Wittgenstein. For, on the picture suggested in NN, objective reference is effected (and secured) not by conscious states of the speaker (like his beliefs) on the one hand, nor by a direct (cognitive) relation he has to the world, on the other, but, basically, by his belonging to a speech community. It is this belonging to a speech community, implied by both (a) and (b), which secures the objective reference of the speaker's words. NN may seem to be worlds apart from the book on Wittgenstein, and as far as I know Kripke has not suggested a connection between them, but this seems to me to be an important link between these celebrated monographs. (See however, SRSR note 20).

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[6] See, for instance, N. Salmon, (1981, ch. 1; H. Wettstein "Has Semantics Rested On a Mistake?", JP 1986, pp.185-209.

Kripke's Modal Notion of Meaning

Although much of Kripke's discussion in NN is concerned with reference-determination, there is a novel and distinctly non-Fregean conception of meaning proposed in NN. Kripke is surprisingly sparse on this.[7] The notion of the meaning (sense) of a name is hardly mentioned in NN except for the above critical remarks about conflating synonymy and reference-determination.

Indeed, he is so sparse on the subject that many scholars ascribe to him the view that proper names have no meaning (sense) (M. Dummett and N. Salmon are notable examples). Although some formulations of Kripke's may support such a view, I believe that ascribing it to him is wrong or, at least, overstated. In fact, the theses that proper names are rigid designators, that they are not synonymous with descriptions, as well as the thesis about the way their actual reference is determined (the causal chain theory) are all theses about their meaning, or sense (cf. pp. 33; 39; 55; 56; 57; 59; we shall come back to this). In another place, Kripke elaborates a bit more on the notion of the meaning of words:

> The notion of what words can mean in the language is semantical: it is given by the conventions of our language. What they mean, on a given occasion is determined, on a given occasion, by these conventions, together with the intentions of the speaker and various contextual features […] If a speaker has a designator in his idiolect, certain conventions of his idiolect (given various facts about the world) determine the referent in the idiolect: that I call the *semantic referent* of the designator(SRSR, p. 14).

(He is clear in this paper that by "semantic" he means "truth-conditional.")

Kripke is usually credited with advancing an effective critique of what he regards to be the Fregean conception of how reference is determined. He is also credited with proposing an alternative picture of the determination of the reference of proper names. He hardly deals in NN with the question of what reference is, what the very ascription of a reference to a word amounts to, and may be regarded as accepting the essentials of the Fregean answer to this question, namely, that the reference of a

[7] Surprisingly, "meaning" does not appear in the index of the book, though, "theory of meaning" occurs under other entries, such as "description theory of proper names".

term is its contribution to determining the truth-value of sentences containing it.[8] Kripke is then credited with making the distinction mentioned above between reference-determination and meaning-determination and with pointing out the Fregean conflation of these. On many of these points Kripke was criticized by Fregean scholars of misrepresenting Frege's views, which, when properly understood, allow, with slight modifications, for most of Kripke's right observations. I have voiced doubts of this sort myself, but I won't go into these here.

But beyond all these I think that Kripke suggested in NN a novel, distinctly non-Fregean conception of meaning, a conception that underlies the main tenor of argument of that work, which I would call *a modal conception of meaning*. Let me quote from the Preface to NN:

> Consider:
>
> (1) Aristotle was fond of dogs.
>
> A proper understanding of this statement involves an understanding both of the (extensionally correct) conditions under which it is in fact true, *and* of the conditions under which a counterfactual course of history, resembling the actual course in some respects but not in others, would be correctly (partially) described by (1). (NN p. 6)

This is couched in terms of understanding, not of meaning. But I think we should not persist in making the distinction, and assume that these two are correlative notions. Again, in explaining the notion (or rule) of rigidity, Kripke says that it is

> "the rule that there is a single individual and a single property such that, with respect to every counterfactual situation, the truth conditions of the proposition are the possession of the property by that individual, in that situation." (p. 10)

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[8] In chapter 5 of my book (Bar-Elli 1996) I expanded on the distinction between "the essential problem" of what meaning (reference or sense) is, namely what the very ascription of meaning to a word amounts to, and "the determination problem" of how the meaning of a word (or a word of a particular kind) is determined. I proposed the distinction in accounting for the relationships between the context principle and compositionality.

This is a conception of meaning and understanding that dominates the course of argument in NN. It is, I believe, a novel one and distinct from a simple truth-conditional view of meaning. On this view the meaning of a sentence like (1) is stated by the conditions (in the actual world) in which it is true and those in which it is false. We don't have to consider its truth conditions in counterfactual situations or possible courses of the world, as in Kripke's conception. Granting its novelty, what is its basis? Why should one, in order to understand a statement, consider its truth conditions in counterfactual situations, and know which counterfactual course of history would be correctly described by it? Kripke evidently thinks that this is not a stipulation concerning an artificial notion of meaning (in the formal semantics of systems of modal logic), but an explicative condition required by our intuitive notion of understanding. Kripke would agree, I suppose, with the view, championed by Dummett and ascribed by him to Frege, that the notion of meaning concerned should serve in an explanation of understanding. But unlike Frege (and Dummett), he thinks that this notion of meaning is *intrinsically modal*, in that the mastery and understanding of simple sentences involve understanding their truth conditions in counterfactual situations.

In thus saying that this notion is intrinsically modal I do not mean to imply that we are concerned here with the notion of the meaning of sentences involving modal operators, or with their role or meaning within complex sentences involving modal operators. Kripke is clear on this, and he emphasizes the point again in the Preface to NN (pp. 11-12), where he also accuses Dummett of misunderstanding this point.[9] The doctrine of rigidity, he says, "is a doctrine about the truth conditions, with respect to counterfactual situations, of (the propositions expressed by) all sentences, including simple sentences" (p. 12). So, in understanding a simple sentence, we have to know its truth conditions *simpliciter* and its truth conditions with respect to counterfactual situations. I mention here what Kripke regards as a misunderstanding on Dummett's part in order to sharpen the novelty of this modal conception of meaning, which is probably what escaped Dummett. Relying on the conventional (i.e. non-modal) conception of meaning, Dummett naturally construed Kripke's insistence on the modal factor in understanding the meaning of a term as understanding the meaning of modal

[9] Dummett probably remained unconvinced, for he repeats virtually the same point, ignoring Kripke's remarks, in his *(1981)*, p. 184, and later in his (1991), p.48.

sentences containing it.[10] But this, on Kripke's view is a mistake. Understanding the truth-conditions of a sentence in a counterfactual situation is required, on Kripke's conception, for understanding any sentence, including simple ones (sentences without modal, or other, operators), with no view to their possible role in modal complexes. In order to understand a simple (non-modal) sentence like (1) one should know not only its actual truth-conditions, but also its truth-conditions in a counterfactual course of history (a course, for instance, where Aristotle was not a philosopher, and had never seen dogs). And it is precisely these truth conditions in a counterfactual situations which render the meaning of (1) different from that, e.g., of its Russellian analysis (p. 9).

It is reasonable to understand this modal conception of meaning in a strong way according to which one should know the truth-conditions of (1) in any counterfactual course of history relevant to the sentence (under some specification of relevance here). Thus, all counterfactual situations which differ from the actual ones concerning Aristotle, and all counterfactual situations which differ from the actual ones concerning dogs (for instance, a situation in which dogs are huge animals tearing to pieces whatever they can get hold of) should be relevant to understanding (1).

One may question my emphasizing the novelty of this conception. For it obviously has its roots in the notion of interpretation in formal semantics and model theory, and as Kripke himself remarked, some of his observations sprang from work in the semantics of modal logic (p. 3). It was moreover brought into philosophy, one may argue, by philosophers like Carnap, who proposed a modal notion of meaning e.g. in *Meaning and Necessity*. But I believe that as a requirement of our general, intuitive conceptions of meaning and of understanding simple (non-modal) sentences in natural language, it is novel.

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[10] I state this with hesitation, for in later writings Dummett seems to endorse the modal notion of meaning. See for instance "Could There Be Unicorns?" in Dummett (1993), p. 335: "Any thesis about the meaning or reference of a word must draw its substance from how we use it or should use it in hypothetical circumstances"; cf. Also 340-341. Whether "hypothetical" includes "counterfactual" is unclear.

This conception has a strong intuitive appeal. One may still wonder about its philosophical basis: what are the general principles governing the notion of meaning and its role in a theory of understanding that warrant this conception?

Our previous remarks seem to suffice for rejecting one possible answer to this query, namely, that the modal conception of meaning is based on the role of words in modal contexts. In understanding a name like "Aristotle" we should know how to evaluate modal contexts in which it occurs. The justification for that might be derived from Frege's context principle: if the meaning of a word is its systematic contribution to all sentences in which it occurs, modal sentences should be considered as well as others. But this, from a Fregean point of view, is a weak defense, and I don't think it plays any role in Kripke's view either. It is weak because, given Frege's compositional conception and his clear hostility to holistic views, it is clear that the meaning of a simple sentence is prior to and independent of the meaning of a complex sentence containing it. Except for operators forming complex sentences, the meaning of a word (e.g. a name) is therefore primarily its contribution to **simple** (atomic) sentences containing it. Hence, the context principle does not provide a justification for the modal conception in the manner suggested.

Alternatively, the modal conception of meaning may have its ultimate grounds in a view Wittgenstein proposed in the *Tractatus* to the effect that understanding a sentence should be construed as independent of the facts, or of knowing the truth of any sentence (cf. 2.0211-2.0212). This is one of the central theses of the *Tractatus*, which Wittgenstein later explicitly rejected. It is unclear to me whether Kripke would want to base his modal conception of meaning (and understanding) on it, but it seems to me a promising avenue.

Up to here I have suggested that NN proposes an important and novel conception of meaning, according to which understanding a sentence, any sentence, involves understanding not only its actual truth conditions, but also its truth conditions in any "relevant" counterfactual situation. It may seem quite natural to regard this modal conception a feature of the non-cognitive attitude towards reference-determination, which, as I said before, dominates NN. For in holding that understanding a sentence involves knowing its truth conditions in counterfactual situations, one is in fact saying that it is independent of the facts we know (for instance, of what we know of Aristotle or dogs). And this may be taken as showing that a cognitive conception of meaning is flawed.

I suggest, however, that this temptation should be resisted. A cognitive conception should not be thus reduced to knowledge of facts: senses and the ways things are given to us are cognitive factors that are not reducible to facts and their knowledge. Moreover, our ability to assess truth conditions in counterfactual situations presumes, and is completely dependent on, our understanding their descriptions, which is itself in the realm of sense and is constrained by the ways things are given to us. Hence, it is not evident that a cognitive conception, in this broad (Fregean) sense, is incompatible with the modal notion of meaning. In the next section I shall examine some of Kripke's specific arguments against the Fregean conception. The point of this examination is not only to see whether Kripke's presentation is faithful to this or that feature of Frege's views, but to demonstrate at what turns Frege's cognitive conception can accord with the modal notion of meaning, by dissociating itself from Kripke's version of the descriptive theory. Hence, Kripke's (convincing) critique of the description theory should not discard a cognitive conception like Frege's.

Kripke's Challenge From a Fregean Perspective

Let us consider a typical example of the kind of arguments advanced in NN against the Fregean conception of the meaning of a name.[11] Consider a simple sentence like (1) "Aristotle was fond of dogs". The argument is based on a line of thought consisting in the following main steps.

1. **The Descriptive Claim**: The sense of a name like "Aristotle" is given by a description, say, "The Greatest philosopher who studied under Plato".
2. **The synonymy assumption**: A name and its sense-giving description are synonymous – have the same sense.
3. **The Kripkean counterfactual move**: Imagine a counterfactual situation in which Aristotle never studied under Plato, and Antistenes was Plato's only student.
4. **The reference question**: To whom does "Aristotle" refer in that counterfactual situation? – Obviously, to Aristotle, not to Antistenes.

[11] Kripke has other arguments, which I shall not discuss here. About one – an epistemic argument – I shall remark later on.

5. **The meaning claim**: The meaning of (1) determines also its truth conditions in the counterfactual situation described (and the meaning of "Aristotle" should be construed accordingly). (cf. NN p. 30; N. Salmon RE pp. 29-30).[12]

This is by now almost a standard move against the Fregean conception, and it is widely considered to be conclusive. But is it? The argument seems to rely on some assumptions, which are worth spelling out:

Assumption (a): The meaning of a sentence is couched in terms of truth conditions.

Assumption (b): In general (perhaps always), the sense of a name is given by a description.

Assumption (c): If "m" gives the meaning (sense) of "n", they are synonymous – have the same sense.

Assumption (d): If two expressions are synonymous they have the same reference in all possible worlds. ("Sense determines reference.")

Assumption (e): Sense is "conceptual"--it is basically a set of properties (or conditions), the satisfaction of which is sufficient and necessary for determining the reference.

Assumption (f): The modal conception of meaning: understanding a sentence involves knowing its truth conditions in counterfactual situations, and understanding a term is knowing its use in such situations.

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[12] A typical example of Kripke's argument on this point is the following. "If 'Aristotle' meant *the man who taught Alexander the Great*, then saying 'Aristotle was a teacher of Alexander the Great' would be a mere tautology. But surely it isn't; it expresses […] something we could discover to be false" (30). One peculiarity of this passage is the use of italics here: why doesn't Kripke put the italicized expression in quotation marks, as he does with "Aristotle"? If to say that "A" means the same as "B" is to state a synonymy, to state that the two expressions have the same meaning, then both expressions should be mentioned – not used. Kripke perhaps uses the italics here as a means for mentioning the meaning of the italicized expresiion (as suggested to me by J. Berg). But then we still lack an argument that this is the only way of giving the meaning of "A", without which the synonymy claim doesn't follow.

I would like to make some comments on these assumptions from a Fregean perspective. The issues are rather intricate and I must be very brief. Let me repeat, however, that the following is written on the assumption of the ambitious reading of NN, according to which it is intended as an attack on a cognitive conception of the meaning of names, and not merely on a descriptive theory of names.

Assumption (a): Sense and Truth Conditions

According to a wide-spread conception, stemming from Frege, meaning is given in terms of truth. According to a strong version of this conception, the meaning of a sentence is its truth-conditions. The meaning of other, sub-sentential expressions is their systematic contribution to the meaning of all, or at least, a distinguished class of the sentences in which they occur. This picture is often ascribed to Frege, who is even acclaimed as being its inventor. Ascribing the general picture to Frege is, I believe, right, but ascribing to him the strong version is, I believe, wrong, but this is not vital for assessing Kripke's argument. Kripke's argument need not rely on this strong version and all it needs is the weaker claim that sentences differing in truth-conditions also differ in sense.

The term "meaning" is notoriously ambiguous between Frege's late sense (*Sinn*) and reference (*Bedeutung*). The aforementioned conception applies, of course, to meaning in the sense of sense. However, Frege, to my knowledge, never speaks of the sense of a sentence as its truth-conditions. By this I don't wish to deny the centrality of the notion of truth in his conception of meaning (and of logic), but this in itself does not mean that the sense of a sentence *is* its truth-conditions. In *Begriffsschrift* (1879) Frege did not yet have the systematic sense/reference distinction and spoke in terms of an undifferentiated notion of content (*Inhalt*). The notion of truth was hardly used at all. Frege there explained his logical operators and logically compound sentences in terms of affirmation (*Bejahung*) and denial (*Verneinung*) of sentences and their possible combinations. Likewise, truth was not used in his later unpublished "Boole's Logical Formula-language…" of 1882. However, the analogues of the affirmation-denial table in "Boole's Logical Calculus…" of 1880 are couched in terms of truth, in something like the modern truth tables (PW, p.11). The notion of truth and its centrality in logic and semantics comes to prominence in Frege's subsequent writings, but, again, not in a way that warrants equating the sense of a sentence with its truth-conditions. He usually talks of the sense of a sentence as the Thought it expresses (SR and many subsequent

writings), where Thought is taken as a primitive, undefinable, and irreducible notion, on a par with Truth. This is particularly true of the sense of simple sentences. The closest Frege comes to the general truth-conditions conception of the sense of a sentence is in section 32 of BL:

> "Every such name of a truth-value expresses a sense, a thought. Namely, by our stipulations it is determined under what conditions the name denotes the True. The sense of this name – the thought – is the thought that these conditions are fulfilled."

There is, however, a marked difference between saying that the sense of a sentence is its truth-conditions, and saying, with Frege, that it is the thought that its truth-conditions are fulfilled. The most obvious difference is that truth-conditions are often construed extensionally: two sentences that are true in exactly the same conditions then have the same truth-conditions. On this view, the way the truth-conditions are given to us, the way we conceive of them, is of no moment. This is evidently not true of Frege's conception, where the senses of two sentences may be radically different even when they have the same truth-conditions. Tautologies are clear examples; statements of identity, as argued in SR, are others.[13] Frege's formulation in terms of the thought that the truth-conditions are fulfilled is evidently not open to this charge, for such thoughts may be different even when the truth-conditions concerned are extensionally the same.

In distinction to many alternative approaches, the Fregean conception is "modest", or non-reductive in its orientation. Sense is explained as the thought that the truth-conditions are satisfied. Thought and its intrinsic opacity or perspectiveness are ineliminable. There can be two different thoughts, with respect to the same truth conditions, that these truth conditions are satisfied. Hence, though sense is couched in terms of truth conditions, Frege would reject the thesis that the sense of a sentence is its truth conditions. However, as stated above, this is not vital for Kripke's argument.

[13] See my (1996), e.g. pp.172-3. The gist of this conception is already stated in section 8 of *Begriffsschrift*, as argued in my (2006).

Assumption (b): The Descriptive Claim

This is a central and a very wide-spread assumption, but its ascription to Frege has been (rightly) rejected by many Frege scholars.[14] I shall therefore not go into it in detail here. This is true of the descriptive theory (of the sense of proper names) in general. It is more emphatically true of a particular strong version of it. The idea of this version is that the description involved is, ultimately, "pure" or "completely universal" in that it does not include names or indexicals like pronouns demonstratives etc. Ascribing the general descriptive view to Frege has very slim basis – a note in SR in which Frege gives an example of a case in which two persons can use the same name (with the same reference) in two different senses. When read in context, this note says nothing like the general descriptive claim. It is implausible that if Frege meant the descriptive claim in this general way he would leave such a central doctrine to what can be implied by a note. But even ther the descriptions in the examples Frege gives are not "pure".

In general, Frege speaks of the sense of a name as a way its reference is presented or given to us, or as a constituent of a thought. The former is the principal way he introduces sense in SR; it recurs, for instance in his 1918 article "Der Gedanke" (65-6; 333 in FR) and in his correspondence (e.g. 80). The latter occurs mainly in BL. There is no reason to suppose that this way can be expressed by such a "pure" description. Fregean sense is often described (mainly by Dummett and many following him) as a "route to the reference", or a way of determining the reference of a term. This has a slim basis in Frege, and has different orientation and overtones from Frege's characterization. Frege's notion of sense – a way something is given to us – is epistemic: it is, as it were, directed from the world to us (as linguistic creatures). The Dummettean characterization has a semantical orientation and goes, as it were, in the opposite direction – from us (or our language) to the world.

Moreover, the descriptive theory, as a particular version of the "route to the reference" conception, has often been conceived in terms of satisfaction – the described object is the one satisfying the description. Satisfying is then construed in terms of a concept applying to (being true of) an object. The relationship between sense (of a name) and reference (object) is therefore construed in this conception in terms of properties or concepts applying to the object – in terms of predication. As an

[14] Beginning perhaps with Dummett's (1973) (e.g. pp. 110, 135).

explication of Frege's view this is definitely wrong (see (e) below). He was explicit that the relationship of sense to reference is not predicative but entirely different.[15] I therefore think that there are good reasons to think that Frege would reject (b).

Assumption (c): Sense-Determination and Synonymy

This assumption is not less problematic from a Fregean perspective. In discussing meaning-determination Kripke assumes that the only way one could "give" or determine the meaning (sense) of an expression is by using a synonymous one. Kripke often talks this way, and passes smoothly from meaning-determination to synonymy determination, as if they were the same. This is a possible and quite widespread conception, but from a Fregean perspective it does not seem to be correct.

The sense of a name is, for Frege, a way its reference is given to us. It is not clear what "giving the sense" of an expression, as it occurs in (b), means here. The clear exception to this is the very special case of giving a stipulative definition (what Frege calls, in "Logic in Mathematics", PW, p. 212, a "constructive (*abauende*) definition"), where an expression which was senseless ("not in use") is given sense by stipulating a definition. This is a case, in which "giving the sense" of an expression is clear, and this is a case in which such a procedure results in stating a synonymy – after stipulating the definition the two expressions have the same sense. But this is a very special case, which is rare outside mathematics and logic, and even there cannot apply to all terms. In the case of many words (names) in a natural language, one does not "give" the sense of an expression by another. Rather, the sense is expressed by using the term to refer to its reference. And even when the sense may be suggested by another expression, as in what Frege calls there "analytical (*zerlegende*) definition", the picture is much more complicated, and they are not synonymous. The criterion of synonymy for Frege is that it must be self-evident, which is rarely, and with respect to proper names almost never, the case. Even his mail logical definitions of the central arithmetical notions of the natural numbers, of the concept of number or of following in a series were not regarded by Frege as self evident, or as resulting in synonymous expressions. "Analytical definitions", or explications, of terms in use, are in general not synonymous to these terms (See PW, pp. 208-212, in particular 210).

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[15] I expanded on this and on a related mistake of regarding senses as objects in my Hebrew book on Frege (2009), chapters 3 and 9.

One can, for example, "give" the meaning (sense) of a name by a particular way of stating what its reference is. Dummett presented Frege's view in a similar manner, using Wittgenstein's distinction between saying and showing. Alternatively, one can explain the meaning of a term, in a Wittgensteinean spirit, by describing the way it is used, or by giving examples of its application, etc. Another approach, perhaps less Fregean and more Davidsonian in spirit was argued by J. McDowell[16], and there are others. All these seem to be compatible with Frege's basic doctrines about sense. And they are all uncommitted to the synonymy condition.

Hence, it is very doubtful whether Frege could accept (c); I think he should not.

Assumption (d): Reference in Possible Worlds

Frege, of course, did not think of sense in terms of possible worlds or counterfactual situations, and it is hard to know what he would say on this. But insofar as this thesis expresses the Fregean doctrine that sense determines reference – a sense has determinate reference in each possible world (waiving aside for the moment indexicals, which pose a special problem), it is quite Fregean in spirit.

Assumption (e): Sense as a Property

Thesis (e) is again objectionable from a Fregean perspective. Properties, for Frege, are concepts – the references of predicates: "I call the concepts under which an object falls its properties" (CO 51, 189 in FR). Therefore, clearly, a sense is not a property (concept) or a set of properties. A sense of a name is a way in which its reference is presented or given to us, and Frege was very clear and persistent in distinguishing it from properties of the referent (object). It is also misleading to think of the sense of a term as a condition, satisfaction of which determines the reference:

"Now it is easy to become unclear about this by confounding the division into concepts and objects with the distinction between sense and meaning (reference), so that we run together sense and concept on the one hand and meaning (reference) and object on the other" (PW p. 118; cf. SR 34 (158 in FR), PW 194, letter to Husserl 24.05.1891, 149 in FR).

[16] "The Sense and Reference of a Proper Name", *Mind*, 1977, 159-185, reprinted in M. Platts *Reference Truth and Reality*, Routledge, 1980).

Frege's conception of sense was intrinsically intentionalistic. A sense is not a mediating entity between a term and its reference – it is a way in which the reference is given, a way that is intrinsically connected to its reference. Frege's conception of reference is, in this light, direct; it is not mediated by a conceptual condition that the reference has to satisfy.[17]

Assumption (f): The Modal Conception of Meaning

I have already suggested, that although Frege did not speak of meaning in modal terms of possible worlds and counterfactuals, such a view can be incorporated quite naturally in his cognitivistic conception of meaning.

We thus see that the argument against Frege's conception of meaning is inconclusive, from a Fregean point of view, independently of the modal conception of meaning. In other words, a Fregean may accept this modal conception of meaning, without accepting the argument in its entirety – he may have enough degrees of freedom to incorporate the modal conception of meaning within the general principles of his theory of sense.

Kripke advances another main argument against the descriptive theory, and claims that the theory is wrong not only with regard to the meaning (sense) of names but also with regard to the determination of their reference. This kind of argument is exemplified in detail mainly in the second lecture of NN, and has often been referred to in the literature as "the epistemic argument". The descriptive theory allegedly claims that the reference of a name is determined by a description – "the so and so" – even though the description need not be giving the meaning (sense) of the name, i.e. the name and the description are not supposed to be synonymous. Kripke advances two arguments against the theory.

(i) It is rarely, if ever, the case that a speaker knows or believes a description, which individuates the referent of a given name. Thus very few users of "Einstein" know or believe individuative descriptions about him (see e.g. pp. 80-2). Hence, their competent use of the name cannot be conditioned on such knowledge.

(ii) It is *a fortiori* not the case that even if the identity between a name and a description is in fact true, it is known a priori, as one could expect it to be if

[17] I have elaborated on this conception of sense and on its implications in my book (1996), esp. chs. 1 and 7.

> the referent of the name were determined by the description (see e.g. pp. 87-90). In other words, even if the description in fact applies to the referent of the name, we can easily conceive that it might turn out not to be so. And even if we strongly believe such an identity, we can easily imagine ourselves to be proved, after all, wrong.

There are two remarks I wish to make about these arguments, and, since I am concerned in this paper mainly with the notion of meaning (not of reference) I shall be very brief here.

First, few people, I believe, would deny (i). It is usually handled by some version of what Putnam called "the division of linguistic labor" (and to which Kripke himself hints in NN). But this seems to suggest that when the speech community is considered as a whole, and when we consider the meaning of a name in a language, not in this or that idiolect, Kripke's counterexamples lose much of their force. For in the community at large some identifying descriptions of the reference are known.

As to (ii), why should we know a priori of a description that it is true of a referent?[18] Well, if the description fixes the referent (as the reference of a term), then it seems we should know a priori that it is true of the referent. If this were the standard or typical way a reference is given or is being "fixed", the argument would have much force. When considered as a thesis against Frege's view, much depends here on how one understands "sense" or "way of being given" or "a mode of presentation" (all translations of Frege's *Art des Gegebenseins* and correolated expressions) in these contexts. Many scholars, including notably Dummett, construe this notion as "a way of determining the referent", or "a route to the reference", something like a sort of an algorithm that one can follow to get to the referent. A descriptive theory of reference-determination fits in quite naturally here. In some very special contexts this picture of sense may be adequate and present Frege's views fairly, but, as noted above, as a general picture of the senses of names, and of the Fregean notion of "mode of presentation", it is I believe, wrong and misleading. Mode of presentation of an object, a way it presents itself to us, as understood by Frege, need not lend itself to a non-circular descriptive articulation. Debarred of this picture, there is no reason to

[18] I leave here aside the difference between the notion of apriority assumed by Kripke (which may be quite the standard one today) and Frege's notion, which has to do only with justification or justifiability (I have expanded on this in Bar-Elli, 2010).

expect a non-circular description to be a priori true of the referent. So here again, strong as the Kripkean arguments may be against a descriptive theory, conjoined with the above algorithmic picture of "determining the reference", they lose much of their force when directed against a cognitive approach, which is dissociated from these, as I believe Frege's view was.

To sum up, although its ultimate philosophical grounds may be unclear, NN makes a strong case for an important, and quite novel, modal feature of the notion of meaning. But this conception, in itself, doesn't tell against the basic principles of Frege's notion of sense. Rather, it can be incorporated within the framework of these principles. As noted in the beginning, NN may be read narrowly, as launching an attack on the descriptive conception of names. This of course is important enough, and from this perspective the question of whether Frege actually held such a position is of minor significance. NN may be read, however, more broadly as having a more ambitious aim: to discard any cognitive conception of meaning and of reference-determination. It is under this more ambitious reading that NN is rightly regarded as a landmark in the philosophy of language, and this, I believe, is the way many people have read it. For people (like myself), who believe that Frege founded and paved a way for a viable cognitive conception of meaning, an argument against such a conception cannot disregard Frege's actual views (even if amended here there). If the arguments in NN are not directed against them as they are, their sting against a cognitive conception of sense is somewhat blunted.

This may hold well, however, also if we disregard Frege's actual views. It is enough if a non-descriptive, cognitive conception of sense and of reference-determination is viable, if it can be based on the notion of a way of being given (for objects and other types of referents). For any such view the Kripkean arguments may remain effective against a descriptive conception of sense and of reference-determination, but much less so against the other sort of cognitive approach to these notions.

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15. "Begriff un Gegenstand" (1892), tr. P. Geach as "Concept and Object", in TPF (also in FR 181-193) – CO.
16. 'On Sense and Reference', tr. M. Black in TPF (also in FR 151-171) – SR.
17. "Gedankegefuege" (1923), tr. P. Geach as "Compound Thoughts", in CP 390-406.

18. "On the aim of *Conceptual Notation*", in CN
19. "Logic", in PW.
20. "My Basic Logical Insights", in PW
21. "Notes for L. Darmstaedter" in PW 253-258.
22. 'Logic in Mathematics' in PW 203-250 - LM
23. "Thought" (tr. P. Geach, in FR 325-345) -T

Often used Abbreviations for Frege's Writings:

BL for [3]	FA for [2]	PW for [5]
BLC [12]	FC for [14]	SR for [13]
BSfor [1]	FG for [4]	T for [23]
CN for [9]	FR for [11]	TPF for [8]
CO for [15]	LM for [22]	
CP for [10]	PMC for [7]	

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Other Abbreviations:

FPL - for Dummett (1981a)
IF - for Dummett (1981b)
LK - for Russell (1956)
NN - for Kripke (1972)
OD - for Russell (1905)
PM - for Russell (1910)
PoM - for Russell (1903)
TLP - for Wittgenstein (1974)